Problems for the Middle School

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Since our first issue in July 2012, we have been carrying problems for the Middle School Section in Problem Corner. We hope that you have enjoyed attempting and solving them. We are taking a brief hiatus with this issue and departing from our usual format to provide all those who would like to improve their problem solving skills at this level with a Handy Reference Sheet for Middle Problems.

As we said in our very first issue, *problems are the life blood of mathematics*. Here is a transfusion: a quick boost for your problem solving cells. We have grouped them according to topic and provided a few problems from past issues which were solved with the help of these facts. In the reference sheet for this issue we have focused on **parity** and **divisibility rules**.

Parity

P1. Every even integer can be written as $2k$, where k is an integer.

P2. Every odd integer can be written as $2k + 1$, where k is an integer.

P3. The square of every even integer is a multiple of 4 and can be written as $0 \pmod 4$.

P4. The square of every odd integer is 1 more than a multiple of 4 and can be written as $1 \pmod 4$.

P5. The sum of the squares of two odd numbers cannot be a perfect square.

P6. When a and b are both odd or both even, the difference of their squares is always a multiple of 4.

P7. For any two integers a and b, the integers $a + b$ and $a - b$ have the same parity, i.e., are either both odd or both even.

Divisibility

These are in addition to the commonly used divisibility rules (by 4, 9 and 11).

D1. The remainder that a number leaves on division by 3 is equal to the remainder that its sum of digits leaves on division by 3.

D2. If k is an odd positive integer, then $a^k + b^k$ is divisible by $a + b$.

D3. Every perfect square is either a multiple of 3 or 1 more than a multiple of 3 and can be written as 0 $(\bmod\ 3)$ or 1 $(\bmod\ 3)$.

D4. Every perfect square is either a multiple of 5 or 1 more or 4 more than a multiple of 5 and can be written as 0 $(\bmod\ 5)$, 1 $(\bmod\ 5)$ or 4 $(\bmod\ 5)$.

As a first exercise, you could try to prove P3 to P7 and D3 and D4 using P1 and P2.

Pythagorean triplets

A *Pythagorean triplet* is a triple (a, b, c) of positive integers such that $a^2 + b^2 = c^2$. Examples: $(3, 4, 5)$, $(6, 8, 10)$, $(5, 12, 13)$. The commonly used short form for a Pythagorean triplet is **PT**.

A PT (a, b, c) is called *primitive* if $\text{GCD}(a, b, c) = 1$. The commonly used short form for a primitive Pythagorean triplet is **PPT**.

Now try to prove these facts about PTs using the given rules.

PT1. It is not possible to have a PT in which exactly two of the numbers are even.

PT2. It is not possible to have a PT in which just two of the numbers are multiples of 3.

PT3. It is not possible to have a PT in which none of the numbers are even.

PT4. It is not possible to have a PT in which none of the numbers are multiples of 3.

Some problems from past issues

Here are some problems from past issues which you can solve using the given facts.

II-3 M.2. It is easy to find a pair of perfect squares that differ by 2013; for example, $47^2 - 14^2 = 2013$.

Now that the new year (2014) is close upon us, we ask: Can you find a pair of perfect squares that differ by exactly 2014?

III-2 M.1. What is the least multiple of 9 which has no odd digits?

III-3-M. 4. Find the digits A and B if the product $2AA \times 3B5$ is a multiple of 12. (Find all the possibilities.)

IV-2 M.2. The sum of the digits of a natural number n is 2015. Can n be a perfect square?

SOLUTIONS OF PROBLEMS IN ISSUE-V-2 (JULY 2016)

The problems studied below are all based on the fundamental concept of divisibility. They require for their solution only a basic understanding of the rules of divisibility.

Solution to problem V-2-M.1

What is the largest prime divisor of every three-digit number with three identical non-zero digits?

Any three-digit number with three identical digits is a multiple of 111. Hence it suffices to focus attention on the number 111. The prime factorisation of this number is $111 = 3 \times 37$. Hence 37 is a prime factor of every number with three identical digits. Therefore the required answer is 37.

Solution to problem V-2-M.2

Given any four distinct integers a, b, c, d, show that the product
$k = (a - b)(a - c)(a - d)(b - c)(b - d)(c - d)$ is divisible by 12.

An integer is divisible by 12 if and only if it is divisible by both 3 and 4. So let us prove that the product k is divisible by both 3 and 4. We now make use of the well-known pigeonhole principle.

Divide the numbers a, b, c, d by 3; each division must leave remainder 0, 1 or 2. There being just three possible remainders, some two of the numbers a, b, c, d must leave the same remainder under division by 3. For these two numbers, their difference is a multiple of 3. Hence k is a multiple of 3.

Next, suppose that two of the four integers a, b, c, d are even and two are odd. Then the differences for both the pairs are even, so k is divisible by 4. If three of the four integers a, b, c, d are even and just one is odd, then three of the differences between the numbers are even, hence k is divisible by 8. The argument is similar when three of the integers a, b, c, d are odd and just one is even. The case when all of the integers a, b, c, d are even (or all four are odd) is trivial.

Editor's remark. The above proposition cannot be strengthened. That is, the 12 in the proposition cannot be replaced by any larger number. To see this, it suffices to consider the four numbers $1, 2, 3, 4$. For this selection, we get $k = 12$. This immediately shows why 12 cannot be replaced by any higher number.

Solution to problem V-2-M.3

Let n be a natural number, and let $d(n)$ denote the sum of the digits of n. Show that if $d(n) = d(3n)$, then 9 divides n. Show that the converse statement is false.

We make repeated use of the tests for divisibility by 3 and 9, namely: a positive integer is divisible by 3 if and only if the sum of its digits is divisible by 3; and likewise for divisibility by 9. We also make repeated use of the notation $a \mid b$ which means that a is a divisor of b.

Assume that $d(n) = d(3n)$. Since $3 \mid 3n$, it follows that $3 \mid d(3n)$. Since $d(n) = d(3n)$, it follows that $3 \mid d(n)$, and therefore that $3 \mid n$. This implies that $9 \mid 3n$, which implies that $9 \mid d(3n)$, and this in turn implies that $9 \mid d(n)$. Now recalling the test for divisibility by 9, we see that $9 \mid n$.

The converse statement is false; e.g., take $n = 126$. Then $9 \mid n$, $3n = 378$, $d(n) = 9$, $d(3n) = 18$, so $d(n) \neq d(3n)$.

Solution to problem V-2-M.4

Let n be an arbitrary positive integer. Show that: (a) $n^5 - n$ is divisible by 5; (b) $n^7 - n$ is divisible by 7; (c) $n^9 - n$ is not necessarily divisible by 9.

(a) We shall use the method of mathematical induction. Let f be the function defined as follows: $f(n) = n^5 - n$. We must prove that $f(n)$ is divisible by 5 for all positive integers n. We observe the following: $f(1) = 0; f(2) = 30; f(3) = 240; f(4) = 1020; f(5) = 3120$. All these numbers are divisible by 5. Next we show the inductive step: namely, if $f(n)$ is divisible by 5, then so is $f(n + 5)$. For this, we only need to simplify the expression for $f(n + 5) - f(n)$. We find that:

$$f(n + 5) - f(n) = 25n^4 + 250n^3 + 1250n^2 + 3125n + 3120,$$

and this is clearly a multiple of 5. By the principle of induction, it follows that $f(n)$ is a multiple of 5 for any positive integer n.

(b) The same approach works for the function $g(n) = n^7 - n$. We first verify that $g(n)$ is a multiple of 7 for $n = 1, 2, 3, \ldots, 7$. Next we simplify the expression $g(n + 7) - g(n)$ and verify that each of its coefficients is a multiple of 7; this implies that $g(n + 7) - g(n)$ is always a multiple of 7. These two statements in combination show that $g(n)$ is a multiple of 7 for every positive integer n.

(c) It suffices to observe that $2^9 - 2 = 510$ is not a multiple of 9.

Solution to problem V-2-M.5
Find all positive integers $n > 3$ such that $n^3 - 3$ is divisible by $n - 3$.

We first divide $n^3 - 3$ by $n - 3$; we get:

$$n^3 - 3 = (n - 3) \cdot \left(n^2 + 3n + 9\right) + 24.$$

It follows that if $n - 3$ divides $n^3 - 3$, then $n - 3$ divides 24 as well. Hence $n - 3$ is a divisor of 24, i.e., $n - 3$ can be any of the numbers $-2, -1, 1, 2, 3, 4, 6, 8, 12, 24$. Hence the possible values of n are: $1, 2, 4, 5, 6, 7, 9, 15, 27$.

Solution to problem V-2-M.6
Show that there cannot exist three positive integers $a, b, c > 1$ such that the following three conditions are simultaneously satisfied: $a^2 - 1$ is divisible by b and c; $b^2 - 1$ is divisible by c and a; $c^2 - 1$ is divisible by a and b.

The conditions imply that a, b, c are mutually coprime. To see why, assume that this is not the case. Suppose that a, b have a common factor exceeding 1; then it would not be possible for $a^2 - 1$ to be divisible by b. The same reasoning works for each pair from a, b, c. It follows that a, b, c are mutually coprime. Hence the given conditions may be replaced by the following: $a^2 - 1$ is divisible by bc; $b^2 - 1$ is divisible by ca; $c^2 - 1$ is divisible by ab.

Since a, b, c are mutually coprime and $a, b, c > 1$, it follows that a, b, c are unequal. Without any loss of generality, we may suppose that $a < b < c$. But in this case we would have $a^2 < bc$, and the requirement that $a^2 - 1$ is divisible by bc is not possible. Hence there cannot exist three positive integers all exceeding 1 which satisfy the stated conditions.

Solution to problem V-2-M.7
Using the nine nonzero digits $1, 2, 3, 4, 5, 6, 7, 8, 9$, form a nine-digit number in which each digit occurs exactly once, such that when the digits are removed one at a time starting from the units end (i.e., from the "right side"), the resulting numbers are divisible respectively by $8, 7, 6, 5, 4, 3, 2, 1$. (So if the nine-digit number is $\overline{ABCDEFGHI}$, then we must have:

$$8 \mid \overline{ABCDEFGH}; \qquad 7 \mid \overline{ABCDEFG}; \qquad 6 \mid \overline{ABCDEF}; \qquad 5 \mid \overline{ABCDE};$$

and so on. Here the notation $a \mid b$ means: "a divides b".)

We argue as follows:

- $5 \mid \overline{ABCDE}$, so $E = 5$.

- Divisibility by 4, 6, 8 implies that B, D, F, H are $2, 4, 6, 8$ in some order.

- Similarly, A, C, G, I are $1, 3, 7, 9$ in some order.

- $3 \mid \overline{ABC}$, hence $3 \mid A + B + C$.

- $6 \mid \overline{ABCD5F}$, hence $3 \mid A + B + C + D + 5 + F$; hence $3 \mid D + 5 + F$; hence $D + F$ leaves remainder 1 on division by 3; hence $D + F = 7$, 10 or 13 (since $D + F$ lies between $2 + 4 = 6$ and $6 + 8 = 14$). Since D, F are even, one of them must be 6 and the other one 4.

- $9 \mid \overline{ABCD5FGHI}$, hence $3 \mid A + B + C + D + 5 + F + G + H + I$, hence $3 \mid G + H + I$.

- $4 \mid \overline{ABCD}$, hence $4 \mid \overline{CD}$. As C is odd, and in any multiple of 4 where the tens digit is odd, the units digit is either 2 or 6, it follows that $D = 2$ or 6. Hence $D = 6$ and $F = 4$.

- Hence among B, H, one must be 2 and the other must be 8.

- The number is now $\overline{ABC654GHI}$. Now $8 \mid \overline{ABCDEFGH}$, therefore $8 \mid \overline{4GH}$, therefore $8 \mid \overline{GH}$ (since $8 \mid 400$). Since G is odd, $H = 2$ or 6. But 6 has already been used, so $H = 2$, and the number is $\overline{ABC654G2I}$.

- Since B is even and $B \neq 2, 4, 6$, it follows that $B = 8$. So the number is $\overline{A8C654G2I}$.

- As stated earlier, A, C, G, I are $1, 3, 7, 9$ in some order.

- Since $3 \mid \overline{A8C}$, it follows that $A + C \equiv 1 \pmod 3$, so $\{A, C\} = \{1, 3\}$ or $\{3, 7\}$ or $\{1, 9\}$.

- The requirement of divisibility by 7 means that $7 \mid \overline{A8C654G}$. We now consider the different possibilities for A, C.

- Suppose that $(A, C) = (1, 3)$; then since $1836540 \equiv 6 \pmod 7$, we must have $G \equiv 1 \pmod 7$, i.e., $G = 1$ or 8. However, 1 has been used up, and G is odd. Hence this possibility cannot happen.

- Suppose next that $(A, C) = (3, 1)$; then since $3816540 \equiv 0 \pmod 7$, we must have $G \equiv 0 \pmod 7$, i.e., $G = 7$ (as 0 is not available). Hence the number is 381654729.

- Suppose that $(A, C) = (3, 7)$; then since $3876540 \equiv 3 \pmod 7$, we must have $G \equiv 4 \pmod 7$, i.e., $G = 4$. However, G is odd. Hence this possibility cannot happen.

- Suppose that $(A, C) = (7, 3)$; then since $7836540 \equiv 5 \pmod 7$, we must have $G \equiv 2 \pmod 7$, i.e., $G = 2$ or 9. Since G is odd, we get $G = 9$. This yields the number 783654921. But this fails the test for divisibility by 8; indeed, $78365492 \equiv 4 \pmod 8$.

- Suppose that $(A, C) = (1, 9)$; then since $1896540 \equiv 2 \pmod 7$, we must have $G \equiv 5 \pmod 7$, i.e., $G = 5$. However, 5 has been used up. Hence this possibility cannot happen.

- Suppose that $(A, C) = (9, 1)$; then since $9816540 \equiv 6 \pmod 7$, we must have $G \equiv 1 \pmod 7$, i.e., $G = 1$ or 8. However, 1 has been used up, and 8 is even. Hence this possibility cannot happen.

- *Hence 381654729 is the only number that satisfies all the requirements.*