Two Problem Studies

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In this edition of *Adventures in Problem Solving*, we continue the theme of studying two problems in some detail. Both problems studied here are from training sessions of the famous Tournament of the Towns (see https://en.wikipedia.org/wiki/Tournament_of_the_Towns); both are accessible to students of classes 9 and 10. We state the problems first, so that you have an opportunity to tackle them before seeing the solutions.

Problem 1: Determine all positive integers n for which there exist n consecutive positive integers whose sum is a prime number.

Problem 2: The sum of all terms of a finite arithmetic progression of integers is a power of 2. Prove that the number of terms is also a power of 2.

Solution to Problem 1: *Determine all positive integers n for which there exist n consecutive positive integers whose sum is a prime number.*

Suppose that the sum of the n consecutive positive integers

$$a, \quad a+1, \quad a+2, \quad \ldots, \quad a+n-1$$

is a prime number; here $a \geq 1$ and $n \geq 2$. (The possibility $n = 1$ clearly need

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not be considered, as it leads to a triviality.) The sum s of these n integers, using the standard formula for the sum of an arithmetic progression, is:

$$s = \frac{n(2a + n - 1)}{2}.$$

It is convenient to divide the analysis into two cases, depending upon whether n is odd or even.

- Suppose that n is odd; let $n = 2k + 1$, where $k \geq 1$ (since $n > 1$). We now have:

  $$s = \frac{(2k+1)(2a+2k)}{2} = (2k+1)(a+k).$$

  In the above expression for s, note that both factors are greater than or equal to 2. Hence s cannot be prime in this case.

- Suppose that n is even; let $n = 2k$, where $k \geq 1$. We now have:

  $$s = \frac{2k(2a + 2k - 1)}{2} = k(2a + 2k - 1).$$

  Since $a \geq 1$ and $2k - 1 \geq 1$, it follows that $2a + 2k - 1 \geq 2$. Hence if the product $k(2a + 2k - 1)$ is to be a prime number, it must be that $k = 1$, i.e., $n = 2$. It can obviously happen that the sum of the 2 consecutive integers $a, a+1$ is a prime number. Indeed, every odd prime can be thus expressed; e.g., $7 = 3 + 4$. The only prime number which *cannot* be expressed in this way is 2.

So the answer to the stated problem is $n = 2$.

Solution to Problem 2: *The sum of all terms of a finite arithmetic progression of integers is a power of* 2. *Prove that the number of terms is also a power of* 2.

Let the arithmetic progression have first term a and common difference d, and let it have n terms:

$$a,\ \ a+d,\ \ a+2d,\ \ \ldots,\ \ a+(n-1)d.$$

The sum s of these terms is given by

$$s = \frac{n(2a + (n-1)d)}{2}.$$

As earlier, it is convenient to divide the analysis into two cases, depending upon whether n is even or odd.

- Suppose that n is odd; let $n = 2k + 1$, where $k \geq 1$ (since $n > 1$). We now have:

  $$s = \frac{(2k+1)(2a+2kd)}{2} = (2k+1)(a+kd).$$

  In the above expression, note that s has the odd factor $2k + 1$, which strictly exceeds 1. Hence s cannot be a power of 2 in this case. (**Remark.** The fact that a power of 2 does not possess an odd factor exceeding 1 is a useful one to record in one's 'math memory'.)

- Suppose that n is even; let $n = 2k$, where $k \geq 1$. We now have:

  $$s = \frac{2k(2a + d(2k-1))}{2} = k(2a + d(2k-1)).$$

We are told that s is a power of 2. Hence it must be that the factors k and $(2a + d(2k - 1))$ are both powers of 2. But since $n = 2k$, it follows that n is a power of 2 as well. (**Remark.** The property that the divisors of a power of 2 are all powers of 2 is another useful fact to record in one's math memory.)

Remark. In connection with the second problem, you may wonder whether every power of 2 can be so expressed, i.e., as the sum of two or more terms of an integer arithmetic progression. The answer is: Yes, but in a rather trivial and none-too-exciting manner. The following array should give an idea of how this is done.

$$\begin{aligned} 4 &= 1 + 3, \\ 8 &= 3 + 5, \\ 16 &= 7 + 9, \\ 32 &= 15 + 17, \end{aligned}$$

and so on. Observe that in each case, we have used only 2 terms in the expression. Can "more interesting expressions" be found, in which the number of terms is 4 or 8 or 16 or some higher power of 2? Indeed we can, for example:

$$\begin{aligned} 16 &= 1 + 3 + 5 + 7, \\ 32 &= 2 + 6 + 10 + 14, \\ 64 &= 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15, \\ 128 &= 2 + 6 + 10 + 14 + 18 + 22 + 26 + 30, \end{aligned}$$

and so on. But we leave the exploration of this to the reader.

Two Often Used Facts…

We have come across two often used facts in these solutions:

- A power of 2 does not possess any odd factor exceeding 1. Moreover, if a positive integer does not possess any odd factors exceeding 1, then it must be a power of 2.
- The only divisors of a power of 2 are smaller powers of 2. Such a statement can be madefor the power of any prime number: the only divisors of a prime power are smaller powersof that same prime number.

As noted above, it is useful to store away these two facts in one's math memory.

The COMMUNITY MATHEMATICS CENTRE (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.