Adventures with Triples

Part II

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In Part I of this two-part note, we described the following two problems. Given three positive integers a, b, c, we say that the triple (a, b, c) has the **linear property** if the sum of some two of the three numbers equals the third number (i.e., either $a + b = c$ or $b + c = a$ or $c + a = b$); and we say that the triple has the **triangular property** if the sum of any two of the three numbers exceeds the third number (i.e., $a + b > c$ and $b + c > a$ and $c + a > b$). Next, we fixed an upper limit n, and let a, b, c take all possible positive integer values between 1 and n (i.e., $1 \leq a, b, c \leq n$); we obtained a total of n^3 triples as a result. Then we asked: *How many of these triples possess the linear property?* We went on to answer this question fully.

Now we ask the same question, but about the triangular triples. However, there are two different contexts in which we can ask this question. We could study only **ordered triples**, in which for example, the triples $(2, 4, 3)$ and $(2, 3, 4)$ are considered to be different from each other; or we could study only **unordered triples**, in which triples such as $(2, 3, 4)$ and $(2, 4, 3)$ are considered to be the same (i.e., we do not distinguish between them). In other words, we do not distinguish between two different permutations of the same triple. *In this article, we choose to study the problem concerning unordered triples.* We ask: *How many of these unordered triples possess the triangular property?* We devote Part II of the article to studying this question.

Keywords: *Integers, linear property, triangular property*

Notation

- $S_u(n)$ denotes the set of all unordered integer triples (a, b, c) with $1 \leq a, b, c \leq n$. As we do not distinguish between two different permutations of the same triple, we may as well insist that $1 \leq a \leq b \leq c \leq n$.

- $T_u(n)$ denotes the number of triples in $S_u(n)$ which possess the triangular property. This is equivalent to defining $T_u(n)$ as the number of integer triples (a, b, c) which satisfy the conditions $1 \leq a \leq b \leq c \leq n$ and $a + b > c$.

Counting the unordered triangular triples

To start with, let us enumerate by hand the values of $T_u(n)$ for a few small values of n.

$\mathbf{n = 1}$: Since $S_u(1)$ has just the one triple $(1, 1, 1)$, and this has the triangular property, $T_u(1) = 1$.

$\mathbf{n = 2}$: The triples in $S_u(2)$ which have the triangular property and are not included in the previous list are $(1, 2, 2)$ and $(2, 2, 2)$; hence $T_u(2) = 2 + 1 = 3$.

$\mathbf{n = 3}$: The triples in $S_u(3)$ which have the triangular property and are not included in the previous list are $(1, 3, 3)$, $(2, 2, 3)$, $(2, 3, 3)$ and $(3, 3, 3)$; hence $T_u(3) = 4 + 3 = 7$.

$\mathbf{n = 4}$: The triples in $S_u(4)$ which have the triangular property and are not included in the previous list are $(1, 4, 4)$, $(2, 3, 4)$, $(2, 4, 4)$, $(3, 3, 4)$, $(3, 4, 4)$ and $(4, 4, 4)$; hence $T_u(4) = 6 + 7 = 13$.

$\mathbf{n = 5}$: The triples in $S_u(5)$ which have the triangular property and are not included in the previous list are $(1, 5, 5)$, $(2, 4, 5)$, $(2, 5, 5)$, $(3, 3, 5)$, $(3, 4, 5)$, $(3, 5, 5)$, $(4, 4, 5)$, $(4, 5, 5)$ and $(5, 5, 5)$; hence $T_u(5) = 9 + 13 = 22$.

$\mathbf{n = 6}$: The triples in $S_u(6)$ which have the triangular property and are not included in the previous list are $(1, 6, 6)$, $(2, 5, 6)$, $(2, 6, 6)$, $(3, 4, 6)$, $(3, 5, 6)$, $(3, 6, 6)$, $(4, 4, 6)$, $(4, 5, 6)$, $(4, 6, 6)$, $(5, 5, 6)$, $(5, 6, 6)$ and $(6, 6, 6)$; hence $T_u(6) = 12 + 22 = 34$.

Proceeding thus, step by step, we construct by hand (or a computer) the following table of values of the T function:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
$T_u(n)$	1	3	7	13	22	34	50	70	95	125	161	203	252	$\cdots$

Do you see any obvious pattern in the sequence of values of $T_u(n)$? There is indeed a pattern, but it is somewhat of a challenge to find it.

A general comment on the analysis of sequences. When we are studying a sequence of numbers and feel that there is some underlying pattern which however we are unable to put a finger on, it often helps to study sub-sequences of the given sequence; for example, by listing every second element of the sequence; or by listing every third element of the sequence; and so on. Hidden patterns sometimes get revealed this way. Another well-known technique is to study the sequence of first differences of the given sequence, or the sequence of second differences.

Applying the idea to our sequence. We shall follow both these suggestions in the present case. We list below the values of $T_u(n)$ for odd n and for even n, separately.

n	1	3	5	7	9	11	13	15	17	19	$\cdots$
$T_u(n)$	1	7	22	50	95	161	252	372	525	715	$\cdots$

n	2	4	6	8	10	12	14	16	18	20	$\cdots$
$T_u(n)$	3	13	34	70	125	203	308	444	615	825	$\cdots$

Now we are able to spot a pattern. In the case of the **odd** values of n, the first differences of the sequence of values of $T_u(n)$ are the following:

$$6,\ 15,\ 28,\ 45,\ 66,\ 91,\ 120,\ 153,\ 190,\ \ldots,$$

and in the case of the **even** values of n, the first differences of the sequence of values of $T_u(n)$ are the following:

$$10,\ 21,\ 36,\ 55,\ 78,\ 105,\ 136,\ 171,\ 210,\ \ldots.$$

Weaving the two sequences of first differences together, we obtain the following:

$$6,\ 10,\ 15,\ 21,\ 28,\ 36,\ 45,\ 55,\ 66,\ 78,\ 91,\ 105,\ 120,\ 136,\ 153,\ 171,\ 190,\ 210,\ \ldots,$$

and this is exactly the sequence of triangular numbers (except that the first two terms, 1 and 3, are missing)!

We thus observe the following, where $t(n) = n(n+1)/2$ denotes the n-th triangular number:

$$T_u(1) = t(1),$$
$$T_u(3) = t(1) + t(3),$$
$$T_u(5) = t(1) + t(3) + t(5),$$

and so on. That is, for $n = 1, 2, 3, 4, \ldots$:

$$T_u(2n-1) = t(1) + t(3) + t(5) + \cdots + t(2n-1).$$

The sum on the right side is easily computed:

$$\begin{aligned} & t(1) + t(3) + t(5) + \cdots + t(2n-1) \\ = \ & \frac{1^2+1}{2} + \frac{3^2+3}{2} + \frac{5^2+5}{2} + \cdots + \frac{(2n-1)^2 + (2n-1)}{2} \\ = \ & \frac{1^2+3^2+5^2+\cdots+(2n-1)^2}{2} + \frac{1+3+5+\cdots+(2n-1)}{2} \\ = \ & \frac{n(2n-1)(2n+1)}{6} + \frac{n^2}{2} = \frac{n\left(4n^2+3n-1\right)}{6}. \end{aligned}$$

So, our conjecture **on the basis of numerical evidence** is that for $n = 1, 2, 3, 4, \ldots$:

$$T_u(2n-1) = \frac{n\left(4n^2+3n-1\right)}{6}.$$

For example, take the values $n = 5$ and $n = 7$. We have:

$$\left.\frac{n\left(4n^2+3n-1\right)}{6}\right|_{n=5} = 95 = T_u(9),$$

$$\left.\frac{n\left(4n^2+3n-1\right)}{6}\right|_{n=7} = 252 = T_u(13),$$

thus confirming the conjecture for these two values of n. Our conjecture is therefore the following: for **odd** values of n, i.e., $n = 1, 3, 5, \ldots$,

$$T_u(n) = \frac{(n+1)\left(2n^2+7n+3\right)}{24}.$$

We similarly conjecture that for $n = 1, 2, 3, 4, \ldots$:

$$T_u(2n) = t(2) + t(4) + t(6) + \cdots + t(2n).$$

The sum on the right side is easily computed:

$$\begin{aligned} & t(2) + t(4) + t(6) + \cdots + t(2n) \\ &= \frac{2^2+2}{2} + \frac{4^2+4}{2} + \frac{6^2+6}{2} + \cdots + \frac{(2n)^2+2n}{2} \\ &= \frac{2^2+4^2+6^2+\cdots+(2n)^2}{2} + \frac{2+4+6+\cdots+2n}{2} \\ &= \frac{n(n+1)(2n+1)}{3} + \frac{n(n+1)}{2} = \frac{n(n+1)(4n+5)}{6}. \end{aligned}$$

So, our conjecture **on the basis of numerical evidence** is that for $n = 1, 2, 3, 4, \ldots$:

$$T_u(2n) = \frac{n(n+1)(4n+5)}{6}.$$

For example, take the values $n = 4$ and $n = 6$. We have:

$$\left.\frac{n(n+1)(4n+5)}{6}\right|_{n=4} = 70 = T_u(8),$$

$$\left.\frac{n(n+1)(4n+5)}{6}\right|_{n=6} = 203 = T_u(12),$$

thus confirming the conjecture for these two values of n. Our conjecture is therefore the following: for **even** values of n, i.e., $n = 2, 4, 6, \ldots$,

$$T_u(n) = \frac{n(n+2)(2n+5)}{24}.$$

So our conjecture on the basis of numerical evidence is the following:

$$T_u(n) = \begin{cases} \dfrac{(n+1)\left(2n^2+7n+3\right)}{24} & \text{if } n \text{ is odd,} \\ \dfrac{n(n+2)(2n+5)}{24} & \text{if } n \text{ is even.} \end{cases}$$

Note that all this is on the basis of numerical evidence. Now we must prove the formula.

Formal proof of the conjecture. Let $f(k)$ denote the number of integer triples (a, b, k) where $1 \le a \le b \le k$ and $a + b > k$. Thus, $f(k)$ counts the number of unordered triangular triples whose largest number is k. It should be evident that

$$T_u(n) = f(1) + f(2) + f(3) + \cdots + f(n).$$

So if we are able to find a formula for $f(k)$, then we should be able to find, using summation, a formula for $T_u(n)$. To find a formula for $f(k)$, it turns out to be convenient to consider separately the cases when k is even and when k is odd.

The case when k is even. Let $k = 2j$. We list all possible triangular triples $(a, b, 2j)$ in a systematic way as follows:

b	List of triangular triples	Number of triples
$2j$	$(1, 2j, 2j),\ (2, 2j, 2j),\ (3, 2j, 2j),\ \dots, (2j, 2j, 2j)$	$2j$
$2j-1$	$(2, 2j-1, 2j),\ (3, 2j-1, 2j),\ \dots, (2j-1, 2j-1, 2j)$	$2j-2$
$2j-2$	$(3, 2j-2, 2j),\ (4, 2j-2, 2j),\ \dots, (2j-2, 2j-2, 2j)$	$2j-4$
$\dots$	$\dots\ \dots\ \dots\ \dots$	$\dots$
$j+1$	$(j, j+1, 2j),\ (j+1, j+1, 2j)$	2

Hence the total number of triangular triples in the case when k is even is equal to:

$$2 + 4 + 6 + \cdots + 2j = j(j+1) = \frac{k}{2}\left(\frac{k}{2} + 1\right) = \frac{k(k+2)}{4}.$$

The case when k is odd. Let $k = 2j + 1$. We list all possible triangular triples $(a, b, 2j+1)$ in a systematic way as follows:

b	List of triangular triples	Number of triples
$2j+1$	$(1, 2j+1, 2j+1),\ (2, 2j+1, 2j+1),\ \dots, (2j+1, 2j+1, 2j)$	$2j+1$
$2j$	$(2, 2j, 2j+1),\ (3, 2j, 2j+1),\ \dots, (2j, 2j, 2j+1)$	$2j-1$
$2j-1$	$(3, 2j-1, 2j+1),\ (4, 2j-1, 2j+1),\ \dots, (2j-1, 2j-1, 2j+1)$	$2j-3$
$\dots$	$\dots\ \dots\ \dots\ \dots$	$\dots$
$j+1$	$(j+1, j+1, 2j+1)$	1

Hence the total number of triangular triples in the case when k is odd is equal to:

$$1 + 3 + 5 + \cdots + (2j+1) = (j+1)^2 = \left(\frac{k+1}{2}\right)^2.$$

It follows that the function $f(k)$ has the following formula:

$$f(k) = \begin{cases} \dfrac{k(k+2)}{4} & \text{if } k \text{ is even,} \\ \left(\dfrac{k+1}{2}\right)^2 & \text{if } k \text{ is odd.} \end{cases}$$

We are now in a position to derive the formula we need. Suppose that n is even, say $n = 2r$; then:

$$T_u(n) = \sum_{\substack{i=1 \\ i \text{ odd}}}^{2r-1} f(i) + \sum_{\substack{i=2 \\ i \text{ even}}}^{2r} f(i)$$

$$= \sum_{\substack{i=1 \\ i \text{ odd}}}^{2r-1} \left(\frac{i+1}{2}\right)^2 + \sum_{\substack{i=2 \\ i \text{ even}}}^{2r} \frac{i(i+2)}{4}$$

$$= \frac{r(r+1)(2r+1)}{6} + \frac{r(r+1)(r+2)}{3} \qquad \text{(steps omitted)}$$

$$= \frac{r(r+1)(4r+5)}{6} = \frac{n(n+2)(2n+5)}{24}, \qquad \text{since } n = 2r.$$

We have recovered the formula which we had empirically derived earlier, purely on the basis of numerical evidence.

Next, suppose that n is odd, say $n = 2r - 1$; then:

$$T_u(n) = \sum_{\substack{i=1 \\ i \text{ odd}}}^{2r-1} f(i) + \sum_{\substack{i=2 \\ i \text{ even}}}^{2r-2} f(i)$$

$$= \sum_{\substack{i=1 \\ i \text{ odd}}}^{2r-1} \left(\frac{i+1}{2}\right)^2 + \sum_{\substack{i=2 \\ i \text{ even}}}^{2r-2} \frac{i(i+2)}{4}$$

$$= \frac{r(r+1)(2r+1)}{6} + \frac{r(r+1)(r-1)}{3} \qquad \text{(steps omitted)}$$

$$= \frac{r\,(4r^2+3r-1)}{6} = \frac{(n+1)\,(2n^2+7n+3)}{24}, \qquad \text{since } n = 2r-1.$$

Once again, we have recovered the formula which we had empirically derived earlier, purely on the basis of numerical evidence.

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