At Right Angles

A Resource for School Mathematics

TEACHING INTEGERS

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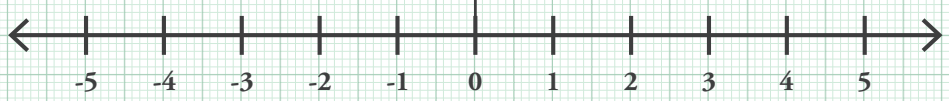

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INTEGERS

Preamble: It is important to state at the outset that manipulatives, which are designed to aid in the understanding of certain concepts, are not meant to be a replacement for the axiomatic development of the subject. Their purpose is essentially to enable students to construct their own mental models for mathematical concepts.

> Introduction of a new topic or a new concept is always a tricky issue. What is a good starting point? Should it begin with the known and then proceed to the new elements? That would make good pedagogic sense.
>
> Or should one probe students' exposure to the topic (applications) through their contact and familiarity with the world and begin there?
>
> Or should one pose questions for which answers do not exist in the past mathematical knowledge that students have acquired?
>
> Or can one create an artificial construct which models a concept?
>
> Perhaps there is no single best way. Every teacher bases such a decision on the knowledge level, age, exposure, readiness and inclination of the students.

The understanding of integers and negative numbers in particular poses certain challenges for many people. One difficulty lies with the fact that the negative sign appears along with a number to indicate an object, a number, but the same negative sign is also used for the operation of subtraction. The other difficulty lies in the way we read, for example: - (-2) as minus negative 2, which is not easily comprehensible. Some of these difficulties are addressed by making the transition from concrete to abstract slowly, as shown in this approach.

'Integers' as a topic is generally introduced in Class 6. Most text books talk about the contextual usage of integers (temperature, seabed, profit and loss, etc.) and use a number line to demonstrate addition and subtraction before progressing to multiplication and division. Some have begun to make use of counters of two colours as well.

I have shared here an approach which I have learnt from Shri P. K. Srinivasan. I have used it for many years successfully with many batches of students and shared it with many teachers in workshop settings. The approach has been received well and appreciated by several teachers who implemented it in their classrooms. It has been an easy task to share this approach in workshops where I could clarify various questions and doubts that were raised. I do hope that in this article, I am able to communicate clearly the various steps involved in this. I also hope that I manage to convey the charm and effectiveness of this approach.

Games have a great appeal for students in general and, in particular, for ten or eleven year olds. This approach begins with a game. It proceeds to teach addition and subtraction using the same materials and helps students to deduce the rules of addition and subtraction. Students find it easy to visualise the problem as it makes use of concrete materials in the initial stages. Also by persisting with the same material and approach for a few days, students are firmly rooted in a particular way of looking at it before encountering it in other forms of representation.

Introduction to Integers through a game

ACTIVITY 1: GAME OF OPPOSITES

Objective: To model the property of integers (opposites cancel each other).

Materials: Red and green buttons (20 of each), a small bag to hold them, a pair of dice.

Rule of the game: Two opposites cancel each other.

Aim: Who has the maximum number of pairs?

Students can be made into two teams.

The players in the first team throw the dice and collect that many buttons from the bag (without seeing the colour of the buttons). Then they lay out the buttons on the floor and remove one pair (a red and green pair) at a time. Then they count the pairs and record the number. Then they put back all the buttons into the bag.

Figure 1

In the same way, the players in the second team throw the dice, collect the required number of buttons, lay them out and count the pairs while removing them, then return the buttons to the bag.

The team with the maximum number of pairs gets a point. If they get the same number of pairs, neither team gets a point.

Figure 2: This shows 3 Pairs

Figure 3: This shows 5 Pairs

The game can be played for several rounds till some team gets all six pairs in a single round.

Students can be exposed to this game for a couple of days before moving on to the next step of teaching.

ACTIVITY 2

Objective: To model addition of integers

Materials: Red and green buttons (10 of each), two dice as shown in Figure 6 (one for colour and one for number), square ruled paper for each child to record results.

Aim: 12 green or red out!

This is a cooperative game. Students record the various steps in square ruled paper as shown in Figure 8.

At this point the teacher can introduce the words positive and negative, that the green button represents a positive and the red button represents a negative. "This is Positive 3." "This is Negative 4." And so on.

Figure 4

Figure 5

The red and green buttons are separated and placed separately on the table.

Figure 6

Figure 7

The first team throws the colour die followed by the number die to pick up the respective buttons and lay them out.

Example: If the colour die shows red and the number die shows 5, the team picks up 5 red buttons.

The second team throws the dice and lays out its buttons.

Example: If the colour die shows green and the number die shows 3, the team picks up 3 green buttons.

The buttons are brought together (combined - 3 greens and 5 reds) and read out by the teacher as: "Now we have positive 3 and negative 5".

Any pairs present are now removed. That leaves behind (2 reds) negative 2.The addition step is recorded as shown, both pictorially and in writing.

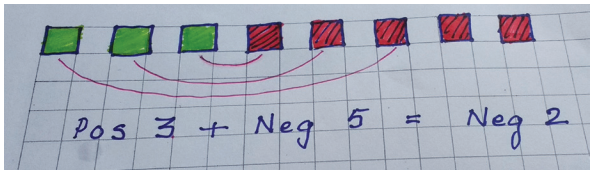

Figure 7a

If at any point the sum becomes (12 green or red buttons) positive 12 or negative 12, the game is over!

Note: This will give rise to all the three possible combinations – positive and positive, negative and negative, positive and negative (red and red, green and green, red and green). Since the numbers involved are small (all less than 7), students may be able to give the result without using the buttons. However, encourage them to record the results as it will help in building their ability to reconstruct it when necessary and aid as a visual memory.

As additional practice, the teacher can also write a few combinations on the board (using numbers less than 10) to get students to do individual work.

Positive 4, Negative 4

Negative 6, Positive 2

Positive 8, Negative 3

Some students who have grasped the concept may be able to give the answer without recourse to pictures. However, they should be encouraged to demonstrate their answers using pictorial models.

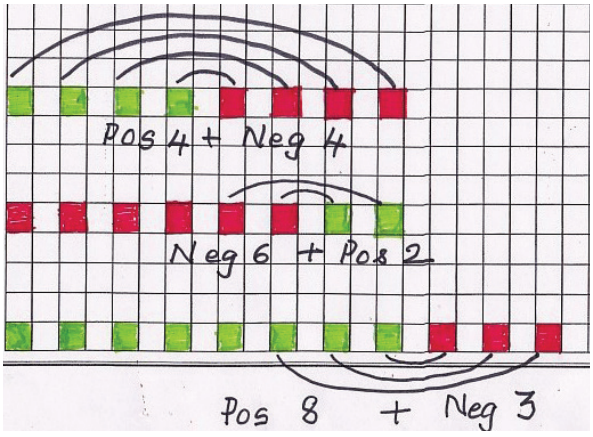

Figure 8

Note: In the initial stages, one uses the words positive and negative. At a later point, once the concept of addition and subtraction is understood thoroughly, the words can be replaced with the signs + and - . This is crucial, as students take time to feel comfortable with the new meaning assigned to + and - . If we can focus initially on the concept and introduce the sign gradually, that would make it easier for the students.

This effectively means that + (plus) and – (minus) will be used initially only to indicate the operations and not as a sign of the number. We use, instead, Pos and Neg to indicate the signs.

ACTIVITY 3

Objective: Oral work in addition of integers.

Aim: Practice in mental visualisation of addition of integers

Once the students have internalised the concept through visual means, they are ready for abstract thinking. The teachers can have oral sessions where larger rounded numbers can be given.

Example: Negative 75, Positive 40 or Positive 200, Negative 120, etc.

ACTIVITY 4

Objective: Rules of addition of integers.

Aim: Observing and deriving the rules of addition of integers

The teacher poses multiple questions for each type of addition situation in integers (Pos+Pos, Neg+Neg, Pos+Neg).

Example: Pos 5 + Pos 4, Pos 2 + Pos 3, Pos 7 + Pos 1.

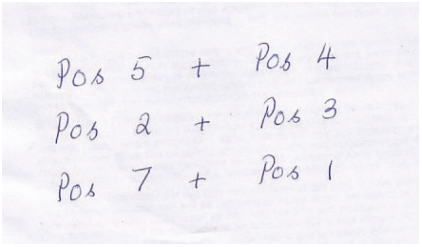

Figure 9

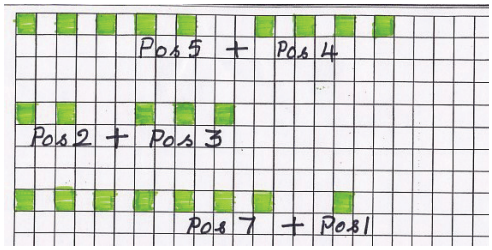

Figure 9a

Students use the materials and record the results on a square paper. They notice that a positive number added to a positive number results in an answer which is positive, and the number is the sum of the given two numbers.

Similarly questions are posed with two negative numbers.

Example: Neg 4 + Neg 6, Neg 2 + Neg 1, Neg 7 + Neg6

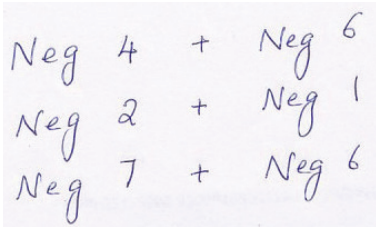

Figure 10

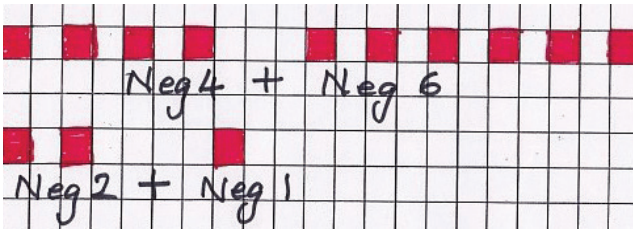

Figure 10a

Students use the materials and record the results on square paper. They notice that a negative number added to a negative number results in an answer which is negative, and the number is the sum of the given two numbers.

The next step is slightly challenging, that is, when we consider a positive and negative combination

Example: Pos 5 + Neg 4, Pos 1 + Neg 7, Neg 5 + Pos 2, Neg 4 + Pos 7

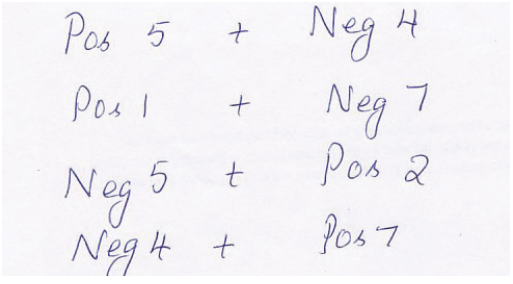

Figure 11

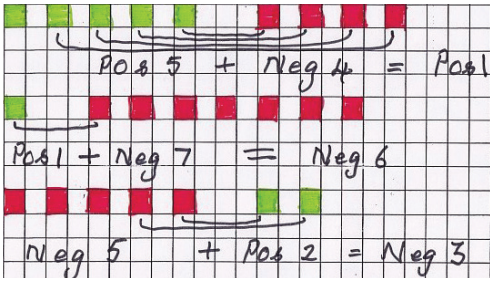

Figure 11a

Again, let them record all the problems pictorially.

Students will need to be guided with appropriate questions to discover the pattern.

The teacher can at first help the students to focus on the numerical part of the answer.

How does the answer connect with the two numbers?

They will be able to see that the number in the answer is the difference of the given two numbers.

How do we know if the result is positive or negative? The teacher points to a specific example and poses the questions, 'Are there more positives here or more negatives?' 'Does that affect the answer?'

Slowly, help the students to formulate the rule for addition of a positive and negative integer.

When a positive and a negative number are to be added, the number in the answer will be the difference of the two numbers and the sign will be that of the integer further from zero.

ACTIVITY 4A

Objective: To model subtraction of integers.

Materials: Red and green buttons (20 of each)

Aim: To be able to do subtraction using the materials.

Subtraction of integers can be classified into two types of situations. I have illustrated the two types using examples.

Situation 1: When there is enough to take away:

Pos 5 – Pos 2 or Neg 7 – Neg 1

Let students pick up buttons corresponding to Pos 5. Ask them to remove Pos 2 from them. There is no difficulty and the result is Pos 3.

Similarly, they pick up buttons corresponding to Neg 7 and remove Neg 1 resulting in Neg 6.

Again, students record these results on square paper.

Figure 12

Situation 2a: When there is not enough to take away and they are of opposite signs

Pos 3 – Neg 2

Ask students to pick up buttons corresponding to Pos 3. They will pick up 3 green buttons.

Now pose the question

'Is there any way of removing Neg 2 from these buttons?'

The initial answer will be a 'NO!'

However, persuade them to think further by giving some hints. 'I can remove Neg 2 from this only if I have Neg 2 here. Is there any way I can have Neg 2 here without changing the value of what I have here?'

Figure 13

Figure 14

'What would be the value of this (of Pos 3) if a pair of opposites is brought in here?'

Students see that it will not alter the value as pairs cancel out each other. Some student may come up with the suggestion that two pairs can be brought in and combined with these buttons.

Figure 14a

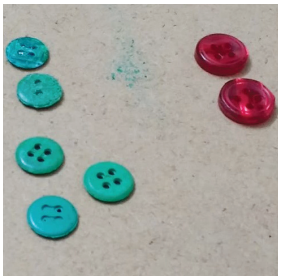

Figure 14b

Now there are Pos 5 and Neg 2 laid out. It is now possible to remove Neg 2.

What is left? Pos 5.

Hence the result for Pos 3 – Neg 2 turns out to be Pos 5.

It is of crucial importance to record this pictorially and write the corresponding steps numerically to understand the logic and the procedure that one follows later.

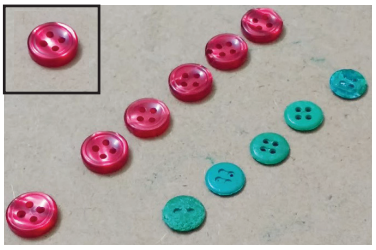

Figure 15a

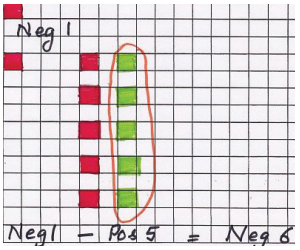

Figure 15b

The same would apply if the problem is Neg 1 – Pos 5.

Student picks up one red button to represent negative one. Now Pos 5 has to be removed.

How many pairs need to be brought in? 5 pairs.

Now there are Neg 6 and Pos 5 laid out. Once Pos 5 is removed there will be Neg 6 left.

Hence the result for Neg 1 – Pos 5 turns out to be Neg 6.

Situation 2b: When there is not enough to take away and they are of same sign

Pos 3 – Pos 5

What should be done here? Students will pick up 3 green buttons to represent Pos 3. How does one remove Pos 5? A few students will quickly see that they can apply the same approach as before, i.e., bring in 5 pairs. Now there are Pos 8 and Neg 5. Pos 5 is now removed.

Three pairs which are present are also removed. What is left?

Hence Pos 3 – Pos 5 turns out to be Neg 2.

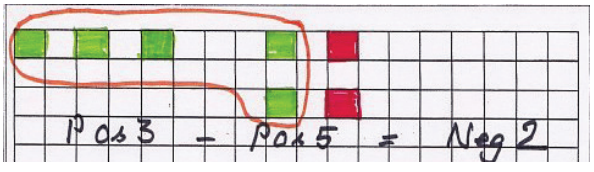

Figure 16

However, some others may suggest that it may be much simpler to bring in only 2 more pairs, as what needs to be removed is Pos 5. In that situation there are Pos 5 and Neg 2 buttons. Once Pos 5 is removed, Neg 2 is left.

So Pos 3 – Pos 5 is Neg 2.

Both approaches are valid; the second one is more efficient.

Students will need to practise doing several of these problems with materials, represent them pictorially and record the steps numerically.

While it is possible to derive rules specifically for different subtraction situations, it becomes cumbersome with too many rules and conditions to remember. Instead, once the students become comfortable with the subtraction concept as shown earlier, it is easier to show that the subtraction of an integer is the same as the addition of its additive inverse. This corresponds to the numerical procedure that we follow normally. This is established in the next activity.

ACTIVITY 5

Objective: To show the link between the concrete method and the numerical procedure

Materials: Square paper

Aim: To understand that subtraction is achieved by adding the additive inverse of the second number

Pose the question Pos 3 – Neg 2.

Let the students record Pos 3 at first in the drawing.

Now comes the crucial step of helping students notice that removing Neg 2 from Pos 3 is effectively leading to adding two pairs (Pos 2 and Neg 2).

Pos 3 is the same as Pos 3 + Pos 2 + Neg2.

Point out that Pos 2 is called the additive inverse of Neg 2, as Pos 2 and Neg 2 make zero.

Now show that Neg 2 can be removed to leave behind Pos 5.

Help them to articulate that subtraction can be achieved by adding the additive inverse of the integer that is to be subtracted.

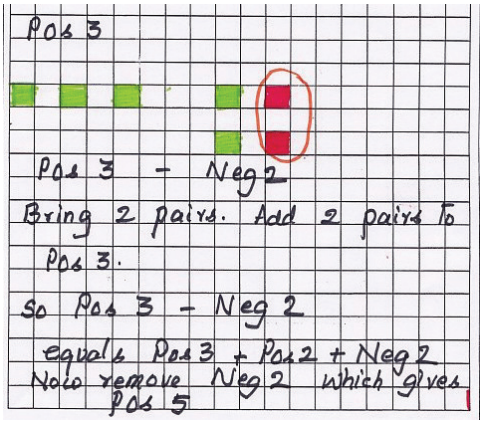

Figure 17

ACTIVITY 6

Objective: To show opposites on a number line and introduce the usage of + and - signs.

Materials: Chalk

Aim: To understand positive and negative numbers and that numbers with a negative sign are less than zero.

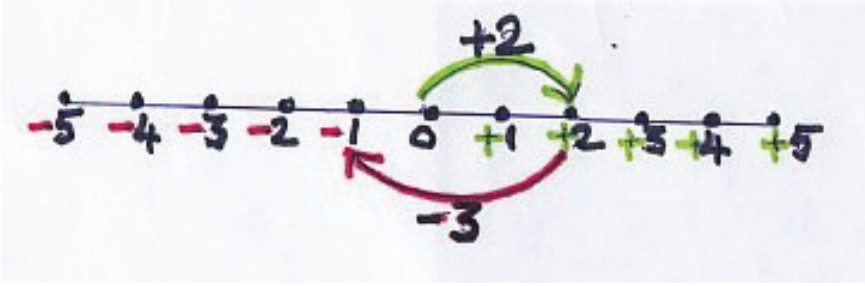

Figure 18

The teacher asks the student to take three steps. The student may take three steps forward without any further questions. The teacher can point out to the child that he or she has not yet specified a direction.

The teacher can now draw a line on the floor and mark 0, 1, 2, 3, 4 and 5. Let the child stand on zero and the teacher repeats the instruction 'take two steps forwards.' Student moves to Point 2. 'Take three steps backwards. ' Student finds that he has to go beyond zero and moves to a new position behind zero.

The teacher now explains that the new position is labelled as negative one. The number line is extended further now on both sides from Positive 10 to Negative 10.

At this point the teacher can introduce the signs for positive and negative, 'Positive' is indicated by a '+' sign and 'Negative' is indicated by a '-' sign'. Numbers are now marked on the number line using the signs.

Now the teacher can mention that '+' is often omitted, i.e., + 2 is written simply as 2.

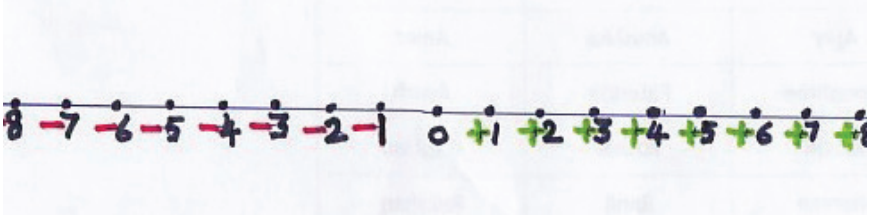

Figure 19

Point out that numbers get smaller as they move away from zero on the left and get bigger as they move away from zero on the right. Also for any two numbers on the number line, the one on the left is smaller than the one on the right. Therefore -3 < -1, although 1 < 3.

Now, using the number line students can compare numbers, determine the bigger and the smaller, successor and predecessor, and derive the rules for comparing numbers.

ACTIVITY 7

Objective: To show usage of integers in the context of temperature

Materials: Temperature chart of some cold places

Aim: To understand the positive and negative numbers in the context of temperature

The teacher discusses the concept of temperature and how it is measured with a thermometer in degrees Centigrade. Temperature chart of their area can be studied together at first. What is the normal temperature on a hot summer day? What would be the night temperature in winter? What would be the temperature in the refrigerator? The teacher can inform the students that water freezes at 0° Centigrade and boils at 100° Centigrade.

Now the teacher can mention that in very cold places the temperature often falls below 0° Centigrade. The students' attention should be drawn to the way the information is recorded, the usage of – sign and the letter C to denote Centigrade.

The teacher and students can discuss the temperature chart of some cold places.

Various comparative statements can now be made based on the drawing.

- Which is the coldest place?
- Which is the warmest place?
- Which places have temperatures below 0°C?
- Which place is 5° higher than -8°C?
- Which place is 2° lower than -10°C?
- How many degrees difference is there between the coldest and the warmest place?

ACTIVITY 8

Objective: To show addition of integers on a number line

Materials: Chalk

Aim: To associate forward movement with positive and backward movement with negative

Till now students have associated positive and negative with positions on the number line. They will also associate adding a positive with a forward movement and adding a negative with a backward movement.

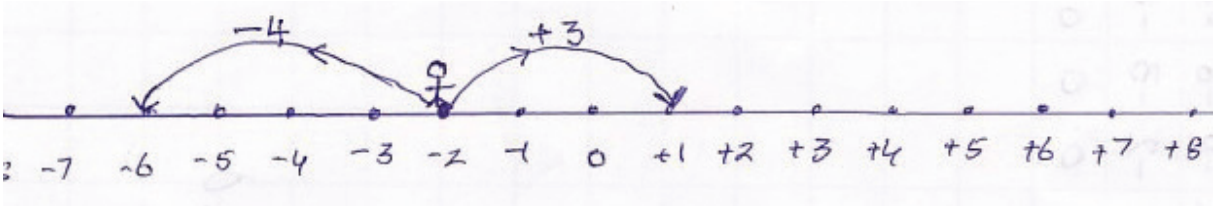

Figure 20

Let a number line be drawn on the ground depicting numbers from -10 to +10. Help the children to understand that adding +2 implies moving forward by two steps from the current position. Adding -3 implies moving backward by 3 steps from the current position.

Students can be asked to demonstrate a problem stated by the teacher.

Example: (-2) + (+3)

Student starts at -2 on the number line and takes 3 steps forward to show addition by +3 and reaches +1 position.

Example: (-2) + (-4)

Student starts at -2 on the number line and takes 4 steps backward to show addition by -4 and reaches -6 position.

Various such problems can be tried out and modelled till all students acquire understanding.

They could also do it the reverse way.

One student demonstrates a problem. Other students have to state the addition fact.

At this point the reader may expect to see subtractions on the number line as the next activity. While it is possible to demonstrate subtractions, procedurally it gets more complex. Subtraction would involve an inverse movement on the number line which becomes difficult for students to comprehend.

I have personally found the red and green button approach adequate. However, if the reader wants to pursue subtractions using a number line, they could consult the NCERT book for Class 6.

Note: It is necessary now to help the students to see the link between the earlier approach that has been followed and the standard methods employed while working with numbers. Teachers may choose to use brackets if it is helpful.

For example:

- Pos 5 + Neg 4 is now written as 5 + (-4) = 1
- Neg 4 + Pos 7 is now written as (-4) + 7 = 3
- Pos 3 – Neg 2 is written as 3 - (-2) = 3 + 2 = 5
- Pos 3 – Pos 5 is written as 3 – 5 = 3 + (-5) = -2

ACTIVITY 9

Objective: To show multiplication of integers using buttons

Materials: Red and green buttons (20 of each)

Aim: To demonstrate multiplication

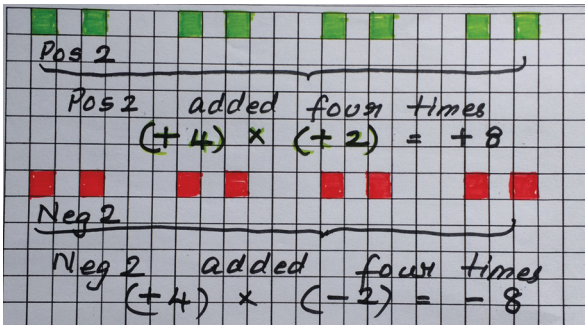

Figure 22

Example: (+4) x (+2)
This is to be read as Pos 2 added four times.
So (+4) x (+2) = +8

Example: (+4) x (-2)
This is to be read as Neg 2 added four times.
So (+4) x (-2) = -8

Example: (-4) x (-2)
This is to be read as Neg 2 subtracted four times.

There is nothing to start with to subtract from. So we bring in the required number of pairs.

Neg 2 is brought in four times along with its opposite Pos 2.

Now we remove (-2) four times. That leaves behind +8.
So (-4) x (-2) = +8

Example: (-4) x (+2)
This is read as Pos 2 subtracted four times.

Again we have nothing to start with. So we bring in two pairs four times.

Now we can remove Pos 2 four times leaving behind Neg 8.

So (-4) x (+2) = -8

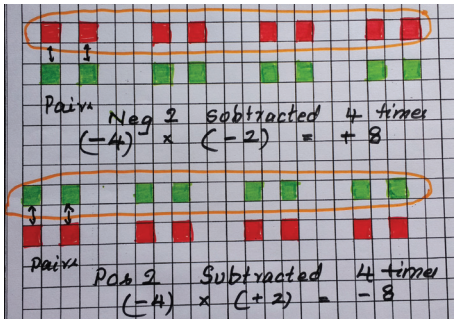

Figure 23

ACTIVITY 10

Objective: To show multiplication of integers

Materials: Square paper

Aim: To derive the rules of multiplication of integers through patterns

Let the students create a table as shown in Figure 21.

In the first step, students are expected to fill in the first quarter of the box.

In the second step, students observe each row (the way the number is decreasing) and continue the pattern beyond zero to complete each row.

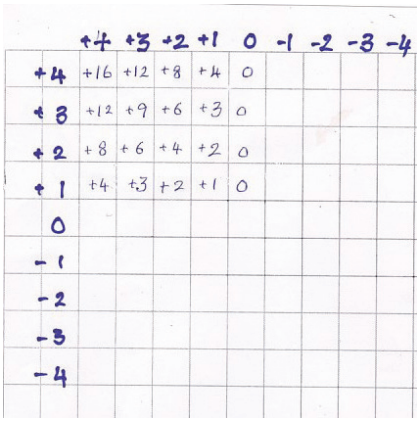

Figure 21

Figure 21a

In the third step, students observe the columns one by one from +4 to 0 and fill in the empty squares.

Now comes the crucial part. The pattern can be continued row wise or column wise. Whichever way it is done, students will see that numbers now start from negative side and increase in steps of 1 or 2 or 3, etc.

The teacher needs to take extra care to point out that now rows and columns start with negative numbers and increase gradually and that the numbers after zero turn positive.

Once the table is filled up and verified by the teacher students can begin to derive rules for multiplication of integers.

Figure 21b

- What happens when a positive number is multiplied by a positive number?
- What happens when a positive number is multiplied by a negative number?
- What happens when a negative number is multiplied by a positive number?
- What happens when a negative number is multiplied by a negative number?

In my experience, I have found the pattern approach simple and satisfactory; children appreciate it.

Activity 10A, below, describes yet another way of demonstrating multiplication (as suggested by PK Srinivasan).

ACTIVITY 10A

Objective: To show multiplication of integers using buttons

Materials: Red and green buttons (20 of each)

Rule: Positive stands for repeating in the same way (repeating the same colour)

Negative stands for repeating the opposite way (repeating the opposite colour)

Aim: To derive the rules of multiplication of integers

Example: Demonstration of (+4) x (+2)

How do we read this? –› This can be read as positive 2 repeated positive 4 times.

How do we understand this? –› Positive 2 is clear, it means 2 green buttons.

Repeating positive 4 times implies repeating four times in the same way.

So 2 green buttons are repeated four times which makes them equal 8 green buttons.

Figure 5

Example: Demonstration of (+2) x (-2)

This can be read as negative 2 repeated positive 2 times.

Repeating positive 2 times implies repeating two times **in the same way.**

Hence (+2) x (-2) equals -4.

Figure 22a

Example: Demonstration of (-2) x (+4)

Here 4 green buttons are repeated twice but in the opposite way (meaning that the colour is changed).

4 green buttons are replaced by 4 red buttons and repeated twice.

Hence (-2) x (+4) equals -8.

Figure 23a

Example: Demonstration of (-2) x (-2)

2 red buttons are repeated twice in the opposite way.

Hence we will get 4 green buttons.

(-2) x (-4) = +8.

ACTIVITY 11

Objective: To show division of integers

Aim: To apply the rules of multiplication of integers to solve division problems

Rules for division flow naturally from the multiplication rules.

Hence, division problems are approached basically by using related multiplication facts.

Example: -10 divided by +5 = -2 (using +5 x -2 = -10)

GAME1: RACE TO THE ENDS!

Objective: To practise addition of integers

Materials: Number line and counters, integer cards (-1 to -10, +1 to +10)

Aim: Who reaches +20 or -20 first?

Players use two differently coloured counters. Both of them start at zero. The integer cards are kept upside down. Each player picks up one integer card and moves that many steps either forward or backwards on the number line based on the card he gets.

Example:

If a player gets -2 first he moves his counter from zero to -2. In the next round if he gets -4 he moves his counter from -2 to -6. Whoever reaches + or – 20 first is the winner.

GAME 2: STRIKE OUT!

Objective: To practise addition and subtraction of integers

Materials: 2 number dice

Aim: To strike out all the numbers from -5 to +12

Players draw a number line marked from -5 to +12. They throw the dice and use the numbers on the dice along with addition or subtraction operation to make up as many numbers as possible between -5 and 12. They strike out the numbers they have been able to make. Each player takes turns to throw the dice and strikes out some numbers. Whoever strikes out all the numbers first is the winner.

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