Features

On the sums of powers of NATURAL NUMBERS Part 2

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Introduction

In Part I of this article, we had considered how to find the formulas for the sums of the squares and the cubes of the first n natural numbers. We had noted that these formulas are routinely encountered in the high school syllabus; they are typically proved using the principle of mathematical induction. In contrast, we had presented a context where the number of ways of carrying out a certain procedure needs to be computed. Two different ways of finding this number were presented. On juxtaposing the results, the desired formulas were obtained as mere corollaries.

Now in Part II of the article, we present a unified approach by which the formula for the sum of the k-th powers of the first n natural numbers can be obtained, for positive integral values of k. The method makes use of a triangular arrangement of numbers which bears a close similarity to the well-known Pascal Triangle.

Power Triangle – The Pascal Triangle

We all know the triangular arrangement of numbers displayed in Figure 1. It is known as the ***Pascal Triangle*** or the *Arithmetic Triangle*, and it yields a simple way of obtaining binomial expansions.

Keywords: Pascal triangle, power triangle, sums of powers

	$r = 0$	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$
$n = 0$	1					
$n = 1$	1	1				
$n = 2$	1	2	1			
$n = 3$	1	3	3	1		
$n = 4$	1	4	6	4	1	
$n = 5$	1	5	10	10	5	1

Figure 1. The first few rows of the Pascal Triangle or Arithmetic Triangle

Its rows are numbered $0, 1, 2, 3, 4, \ldots$, and the numbers in the n-th row give us the coefficients of the successive terms in the expansion of $(a+b)^n$, for $n = 0, 1, 2, 3, \ldots$; i.e., the numbers of the Pascal triangle are the binomial coefficients $\binom{n}{r}$. The rule of formation of the numbers is this:

(i) the 0-th row has just one number, namely, 1;

(ii) the n-th row has $n+1$ numbers;

(iii) the first and last numbers in each row are 1, i.e.,

$$\binom{n}{0} = 1 = \binom{n}{n}, \quad \text{for } n = 0, 1, 2, 3, \ldots;$$

(iv) every other number of the Pascal triangle is given by the sum of the number immediately above it, and the number immediately to the left of that number (this is the well-known property of the Pascal Triangle), i.e.,

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}, \quad \text{for } n = 1, 2, 3, \ldots \text{ and } r = 1, 2, 3, \ldots, n-1.$$

(v) if the element at any position is absent, it is taken to be 0.

The Power Triangle. Now we display another such triangular array. As noted, it bears a certain resemblance with the Pascal Triangle. It is of comparatively recent origin and it helps in finding a formula for the sum of the k-th powers of the first n natural numbers, for any given positive integer k. It is called the *Power Triangle*. Its first few rows are given in Figure 2, and we can extend it indefinitely.

Rules governing the formation of the Power Triangle. Denote the number in row n and column r by $T(n, r)$; here $n = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots, n+1$. Then:

Rule 1: Row n has $n+1$ numbers, $T(n,1)$, $T(n,2)$, $T(n,3)$, …, $T(n, n+1)$. We adopt the convention that $T(n,r) = 0$ if $r < 1$ or if $r > n+1$. (In words: if the element at any position is absent, it is taken to be 0.)

Rule 2: The first number of every row is 1; so $T(n,1) = 1$ for $n = 0, 1, 2, \ldots$.

Rule 3: The numbers in the successive rows of the power triangle are determined recursively as follows: for $n = 1, 2, 3 \ldots$ and $r = 1, 2, 3, \ldots, n+1$,

$$T(n,r) = (r-1) \cdot T(n-1, r-1) + r \cdot T(n-1, r). \tag{1}$$

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$
$n=0$	1					
$n=1$	1	1				
$n=2$	1	3	2			
$n=3$	1	7	12	6		
$n=4$	1	15	50	60	24	
$n=5$	1	31	180	390	360	120

Figure 2. The first few rows of the Power Triangle

Many nice properties of the T-numbers can be derived using the recursive property repeatedly. For example,

$$T(n,2) = 2^n - 1,$$
$$T(n,n+1) = n!\,,$$

for all positive integers n.

Using the Power Triangle

Now we show how this number array can be used to find a formula for the sum of the k-th powers of the first n natural numbers. The formula used is this:

$$1^k + 2^k + \cdots + n^k = \binom{n}{1} \cdot T(k,1) + \binom{n}{2} \cdot T(k,2) + \cdots + \binom{n}{k+1} \cdot T(k,k+1),$$

i.e.,

$$1^k + 2^k + \cdots + n^k = \sum_{r=1}^{k+1} \binom{n}{r} \cdot T(k,r). \tag{2}$$

As simple as that! Now let us see this remarkable formula in action.

Sum to power 0: Here $k=0$; so:

$$1^0 + 2^0 + \cdots + n^0 = \binom{n}{1} \cdot T(0,1) = n \cdot 1 = n. \tag{3}$$

Clearly true!

Sum to power 1: Here $k=1$; so:

$$\begin{aligned} 1^1 + 2^1 + \cdots + n^1 &= \binom{n}{1} \cdot T(1,1) + \binom{n}{2} \cdot T(1,2) \\ &= n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}. \end{aligned} \tag{4}$$

We have recovered the familiar formula.

Sum to power 2: Here $k = 2$; so:

$$
\begin{aligned}
1^2 + 2^2 + \cdots + n^2 &= \binom{n}{1} \cdot T(2,1) + \binom{n}{2} \cdot T(2,2) + \binom{n}{3} \cdot T(2,3) \\
&= n + 3 \cdot \frac{n(n-1)}{2} + 2 \cdot \frac{n(n-1)(n-2)}{6} \\
&= \frac{n(n+1)(2n+1)}{6}, \quad \text{on simplification.}
\end{aligned} \tag{5}
$$

Once again, we have recovered the familiar formula.

Sum to power 3: Here $k = 3$; so

$$
\begin{aligned}
&1^3 + 2^3 + \cdots + n^3 \\
&= \binom{n}{1} \cdot T(3,1) + \binom{n}{2} \cdot T(3,2) + \binom{n}{3} \cdot T(3,3) + \binom{n}{4} \cdot T(3,4) \\
&= n + 7 \cdot \frac{n(n-1)}{2} + 12 \cdot \frac{n(n-1)(n-2)}{6} + 6 \cdot \frac{n(n-1)(n-2)(n-3)}{24} \\
&= \frac{n^2(n+1)^2}{4}, \quad \text{on simplification.}
\end{aligned} \tag{6}
$$

Yet again, we have recovered the known formula. We can continue in this vein indefinitely and produce more and more such formulas.

Closing remark. The power triangle provides a simple and extremely convenient way of recovering the formulas for the sums of the k-th powers of the first n natural numbers, for any given positive integer k.

But note, however, that we have not provided any *proof* of the method. Interested readers may refer to the companion article by Prof K D Joshi, in which a proof has been sketched.

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