ClassRoom

Low Floor High Ceiling

Four-telling Quadrilaterals

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A mathematical investigation is an exciting way for students to learn to develop systematic reasoning and mathematical rigor in a relaxed and easy going manner. Setting up an investigation which achieves this objective is, however, not always easy. In addition, the teacher needs to facilitate the proceedings in such a manner that the student takes ownership of the investigation and develops these skills in a natural manner.

As a natural part of their everyday mathematics work, investigations help students:

- Explore problems in depth
- Find more than one way to solve many of the problems they encounter
- Reason mathematically and develop problem-solving strategies
- Examine and explain mathematical thinking and reasoning
- Communicate their ideas orally and on paper, using "clear and concise" notation
- Represent their thinking using models, diagrams and graphs
- Make connections between mathematical ideas
- Prove their ideas to others
- Develop computational skills – efficiency, accuracy and flexibility
- Choose from a variety of tools and appropriate technology
- Work in a variety of groupings – whole class, individually, in pairs, in small groups

Source: http://www.canalwinchesterschools.org/WTIMP.aspx

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Clearly, these are objectives worth pursuing. Here is an investigation on types of quadrilaterals which, while starting with some simple hands-on activities and documentation of findings, ramps up to a conjecture and finally a proof about isosceles trapeziums. At several points, there are potential **I**nvestigation **Q**uestions which students can diverge to; however, in order to keep a somewhat linear flow, we have indicated these with a ***IQ#**. The interested teacher or student can design several new investigations based on these suggestions.

Part I: Classification of Quadrilaterals

Just as there is a triangle rule, according to which the length of any side of a triangle is greater than the difference of the lengths of the other two sides and less than the sum of the lengths of the other two sides, we have an equivalent quadrilateral rule. Here, the length of any side of a quadrilateral is less than the sum of the lengths of the other three sides.

1. Keeping this rule in mind, how many kinds of quadrilaterals can you make

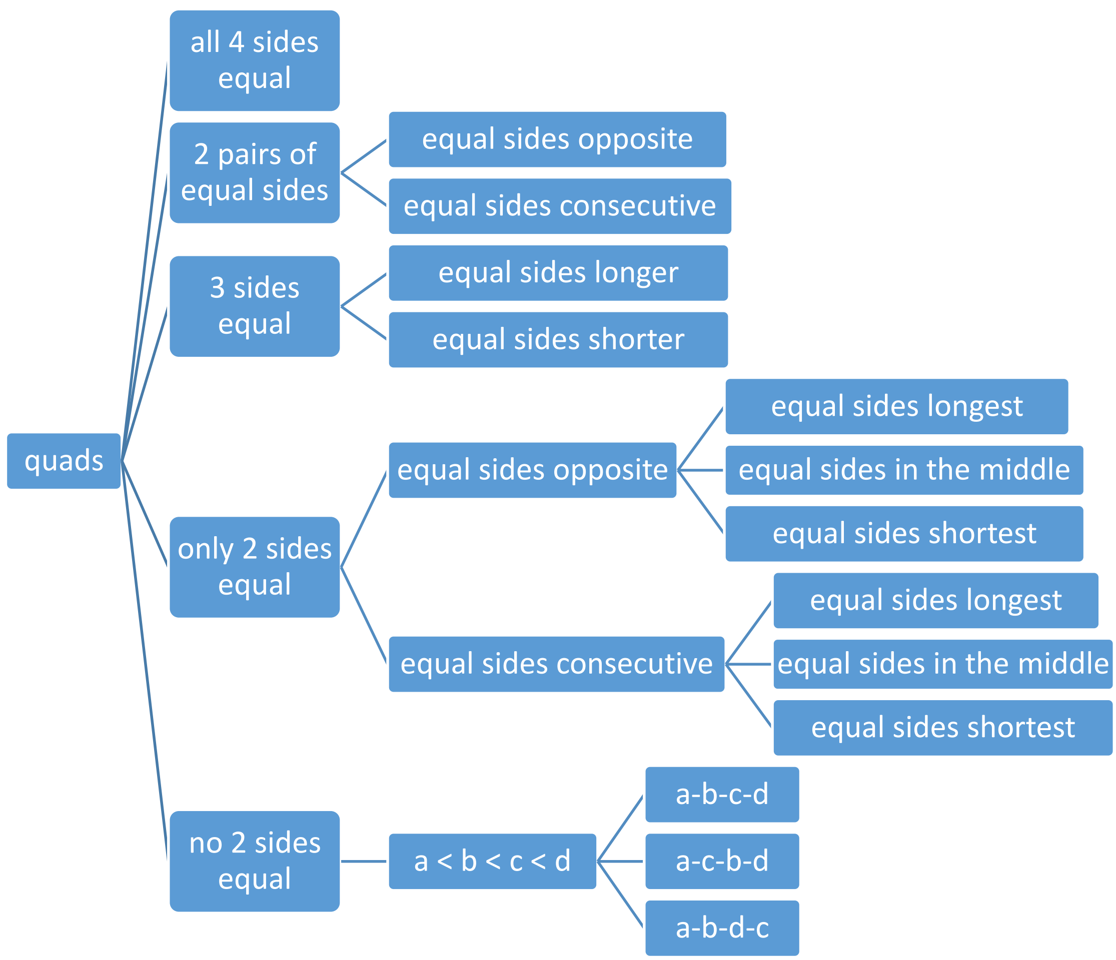
a. With 4 matchsticks **b.** With 8 matchsticks **c.** With 12 matchsticks **d.** With 16 matchsticks

Compile your results in a table as follows (some results have been given, a selection may be provided for students depending on their proficiency in this activity).

***IQ1:** While we have given only feasible quadrilaterals, it may be worthwhile to do a whole class investigation of all the possible side lengths which add up to 4, 8 ,12 or 16 and select those combinations which represent sides of a quadrilateral; do encourage students to write up and share their reasoning process.

	Side combinations	Possible quads
4 matchsticks	1-1-1-1	Rhombi
8 matchsticks (investigate why 1-2-4-1 is not possible)	2-2-2-2	Rhombi
	1-3-1-3	Parallelograms
	1-1-3-3	Kites and darts
12 matchsticks	3-3-3-3	
		Parallelograms
	2-2-4-4	
	2-2-3-5	Isosceles trapezium and general quadrilaterals
		General quadrilaterals with a pair of consecutive equal sides
		General quads with 4 different sides
16 matchsticks		
	⋮	⋮

2. **Try to generalize the pattern and capture it in a tree diagram as follows:**



***IQ2:** Many investigation questions can be designed at this point, using each branch of this tree diagram. Students who know the basic definition of the different kinds of quadrilaterals can see which branches can generate feasible (and not just theoretical) quadrilaterals of a particular kind.

***IQ3:** This is also an opportunity for students to study line symmetry and to classify all these quadrilaterals as those with or without line symmetry.

***IQ4:** For those who want to play with concave and convex quadrilaterals, we suggest straw models to investigate which of these quadrilaterals can be concave and which convex. Instructions for making straw models are given in the box.

To make a straw model for any of the above quads, take two straws of roughly equal length and cut them into four pieces as per specification.

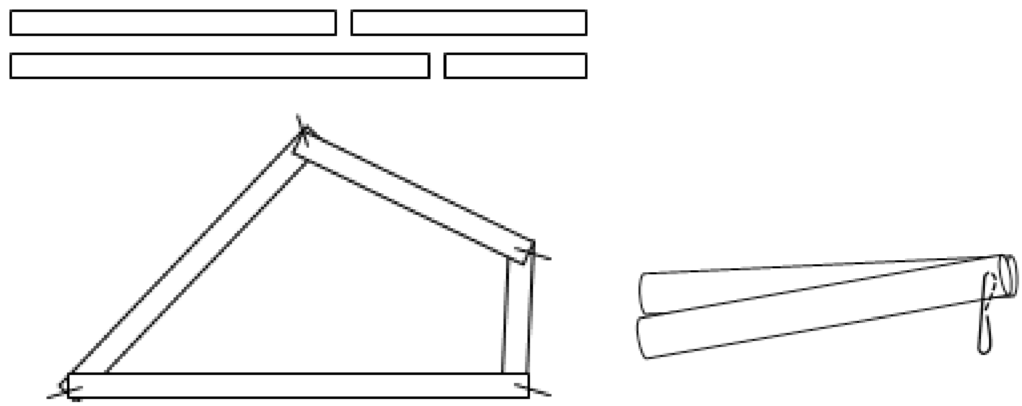


Figure 1.

For example, to make any of the quads with four different sides: cut one straw in roughly equal halves but not exactly so that one part is a bit longer than the other – these are b < c. Cut the second straw so that one part is much longer than the other– these are a < d.

The above cuts and the roughly equal length of the straws ensure that a < b < c < d. Now join them, with a stapler, in any of the given sequences to get the corresponding quad. While stapling, only one tooth of the pin should pass through the straw while the other will remain completely outside. After stapling, the straws should move freely forming a range of angles between them.

3. Which of the above branches can generate isosceles trapeziums?

Teacher's Note:
Students may need to refresh their definition of an isosceles trapezium: In Euclidean geometry, an **isosceles trapezoid** (**isosceles trapezium** in British English) is a convex quadrilateral with a line of symmetry bisecting one pair of opposite sides. It is a special case of a trapezoid. In any isosceles trapezoid two opposite sides (the bases) are parallel, and the two other sides (the legs) are of equal length (properties shared with the parallelogram). The diagonals are also of equal length. The base angles of an isosceles trapezoid are equal in measure (there are in fact two pairs of equal base angles, where one base angle is the supplementary angle of a base angle at the other base). https://en.wikipedia.org/wiki/Isosceles_trapezoid

***IQ5:** Mathematics, and geometry in particular, is rich in equivalent conditions as seen in the above, rather detailed, description. Isosceles trapeziums can provide an opportunity for students to investigate such equivalent conditions. Here is a sample question: 'If a quadrilateral has a line of symmetry bisecting a pair of opposite sides, what other properties does it have?'

***IQ6:** The converse can be more interesting. What is the converse of the above question? What are the minimum conditions to be specified to define an isosceles trapezium?

Apart from the quadrilaterals with four equal sides (in general, rhombi) and the ones with two pairs of equal and opposite sides (in general, parallelograms) in the tree diagram, only 5 types can yield isosceles trapeziums. These are:

i. 3 sides equal with equal sides longer, represented by b-b-b-a

ii. 3 sides equal with equal sides shorter, represented by a-a-a-b

iii. Only 2 sides equal and opposite and shortest, represented by a-b-a-c

iv. Only 2 sides equal and opposite and medium sized, b-a-b-c

v. Only 2 sides equal and opposite and longest, represented by c-a-c-b
Throughout we will use the notation that $a < b < c$

Part II: Moving to trapeziums via parallelograms.

1. **Suppose you are given both the lengths of the adjacent sides of a parallelogram. How many parallelograms can you construct?**

2. **Suppose you are given the lengths of all four sides of a trapezium, how many trapeziums can you draw? Investigate with the following example.**

   Construct a trapezium ABCD with AB = 3cm, BC = 4cm, CD = 5cm, AD = 7cm. Assume AD || BC.

   Hint:
   Construct ΔCDE with CD = 5cm, CE = 3cm and DE = 3cm and extend DE to DA = 7 cm. Through C, draw CB = 4 cm. parallel to DA. How many such trapeziums can there be?

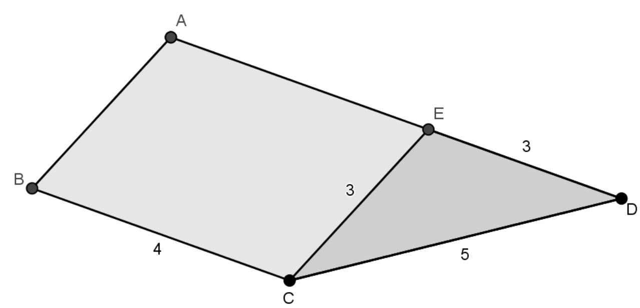


Figure 2. Trapezium with all four side lengths given

Teacher's Note:

Infinitely many parallelograms can be constructed since the angle between the given sides can be anything between 0° and 90°.

However, if the lengths of all 4 sides are given and the sides which are parallel are specified, then only one trapezium satisfies these constraints.

3. **Can you draw a quadrilateral ABCD with the same sides as given in 2 (AB = 3, BC = 4, CD =5, DA = 7) but with AB || CD? Justify.**

Teacher's Note:

Suppose it is possible. Then there is a point E on CD such that CE = AB = 3cm and ED = 2cm. Join AE. Now AB || CD $\Rightarrow$ AB || CE and AB = CE $\Rightarrow$ ABCE is a parallelogram $\Rightarrow$ BC = AE = 4cm. Now in ΔADE, AD = 7cm < AE + ED = 4cm + 2cm = 6cm which is a contradiction! So it is impossible to draw the trapezium in this case.

4. **Now to generalise this: Consider trapezium ABCD such that BC || AD and BC < AD without loss of generality (Note that BC = AD implies ABCD is a parallelogram). Use the above split of trapezium ABCD into parallelogram ABCE and ΔCDE as mentioned in 2.**

Figure 3.

a. If BA ≤ CD without loss of generality, is BA + AD less than, equal to or greater than BC + CD?

b. If BC < AD, BA ≤ CD and BA + AD > BC + CD, is it possible to draw trapezium ABCD with BC || AD?

c. Given, BC < AD, BA ≤ CD and BA + AD > BC + CD, can you draw the trapezium with AB || CD?

d. What do you think happens if AB + BC = CD + DA?

Teacher's Note:

Using triangle inequality for the sides of ΔADE, where AE = BC and using the fact that ABCE is a parallelogram, it can be shown that CD < BA + AD – BC ⇒ BA + AD > BC + CD. Note that this is non-trivial since AD > BC but BA ≤ CD.

The converse is also true. The trapezium can be constructed for BC || AD (follow the hint given in 2) and not for AB || CD (check the Teacher's Note for 3). So for BC < AD and BA ≤ CD, BC || AD if and only if BA + AD > BC + CD.

When AB + BC = CD + DA, the triangle inequality collapses ΔADE into the line segment CD resulting in a collapse of parallelogram ABCE to the line segment BC + CD and we get a degenerate trapezium flattened to a line segment BD = BC + CD = BA + AD.

Part III: Focusing on the isosceles trapeziums.

1. **Consider any isosceles trapezium. Is it cyclic?**
2. **Consider any cyclic trapezium. Is it isosceles?**

Teacher's Note:

Suppose ABCD is an isosceles trapezium with BC || AD and BA = CD. Then ∠D = ∠A. [Hint: Split the trapezium into a parallelogram and a triangle to prove this.]

Assume BC < AD without loss of generality (BC = AD implies that ABCD is a rectangle which is cyclic). So ∠A + ∠C = ∠D + ∠C = 180° since BC || AD. So ∠B + ∠D = 360° – (∠A + ∠C) = 180° ⇒ ABCD is cyclic.

Alternatively, suppose ABCD is a cyclic trapezium i.e. BC || AD and ∠A + ∠C = ∠B + ∠D = 180°. ∠A + ∠C = 180° = ∠C + ∠D since BC || AD. So ∠A = ∠D ⇒ AB = CE = CD i.e. ABCD is isosceles. [The last step can be proved using the same hint as above.]

Therefore, a trapezium is isosceles if and only if it is cyclic.

***IQ7:** If I want to construct an isosceles trapezium what is the minimum amount of data that I need to have? Note: there are multiple possibilities e.g. (i) parallel sides and the distance between them, (ii) parallel sides and a base angle, etc. List as many possibilities as you can.

3. **Consider the isosceles trapezium a-c-b-c (remember that a < b < c). We have just seen that it will be cyclic. Is it possible to inscribe such a trapezium in the minor segment of a circle?**

4. **Determine if the isosceles trapezium a-b-b-b will always be in the major segment.**

Teacher's Note:

In ΔSQR (Figure 4), $SR = b < QR = c \Rightarrow \angle SQR < \angle QSR \Rightarrow \angle SQR$ is acute $\Rightarrow$ SPQR is a major arc

A similar proof can be devised for the trapezium a-b-b-b (Figure 5).

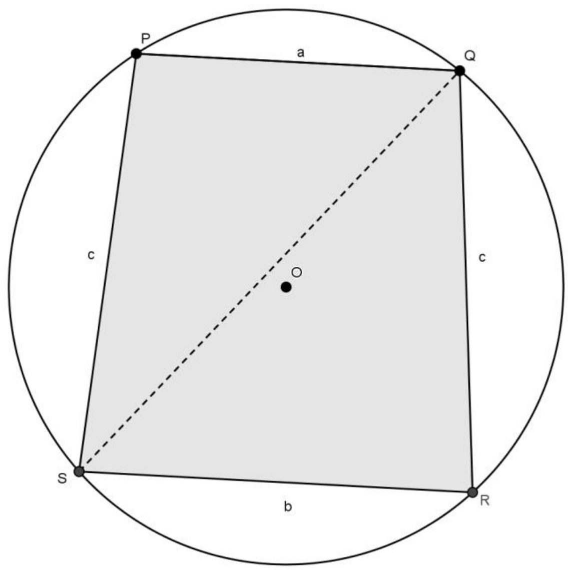


Figure 4. Isosceles trapezium a-c-b-c

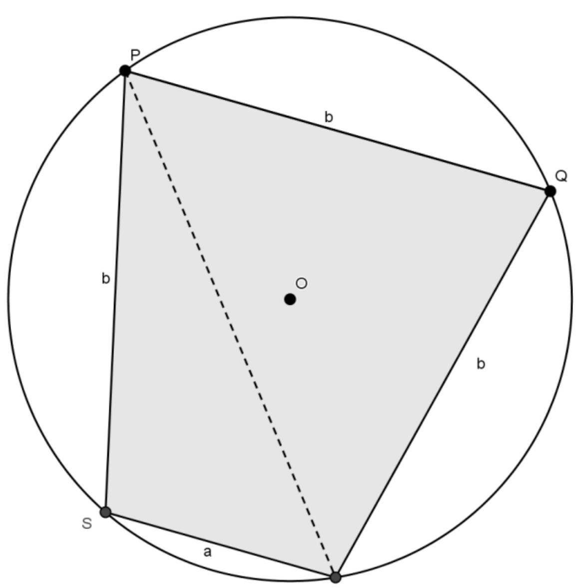


Figure 5. Isosceles trapezium a-b-b-b

5. **Consider the isosceles trapezium a-a-a-b (remember $a < b$).**

Note that $b < 3a$ to satisfy the quadrilateral inequality mentioned at the beginning of the article.

a. Suppose this trapezium has its longest side on the diameter. Prove that $b = 2a$.

b. Find the relation between a and b if this quadrilateral fits

i. in a minor segment

ii. in a major segment

Teacher's Note:

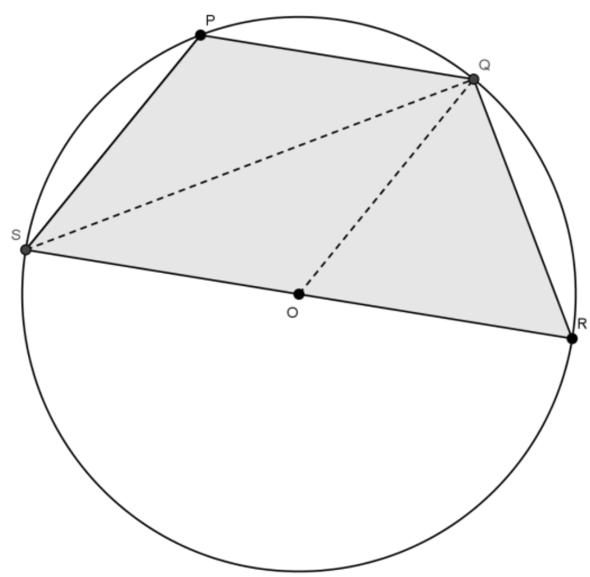


Figure 6. Isosceles trapezium in a semi-circle

$SP = PQ = QR = a$

$SR = b$

$\angle PQS = \angle QSR = \theta$ (alternate angles)

$\angle PQS = \angle PSQ = \theta$ (isosceles triangle)

$\therefore \angle PSR = 2\theta$

$\angle SQR = 90°$ (angle in semi-circle)

$\therefore \angle PQR = 90° + \theta = \angle QPS$

And $\angle PSR = \angle QRS =$

$90° - \theta = 2\theta$ (supplementary angles)

$\therefore 90° - \theta = 2\theta \Rightarrow \theta = 30°$

In ΔQOR, $\angle OQR = \angle QRO = 90° - \theta = 60°$

$\angle QOR = 2\theta = 60^0$

So ΔOQR is equilateral and $\therefore QR = a = OR = b/2$

If PQRS fits in the minor segment, then $\angle SQR > 90°$ ⇒ with lengths $SP = PQ = QR = a$ fixed, S and R should be further apart (resulting in a larger circumcircle) ⇒ $b > 2a$ and similarly if it fits in the major segment, then $b < 2a$.

Similar exploration can be done for the remaining two isosceles trapeziums. We will discuss those in a subsequent article.

Conclusion

Playing with matchsticks and straws can generate a lot of heat. Of the safe variety, thankfully. We hope that you will have fun investigating and writing up your discoveries. Do try these activities on dynamic geometry software such as GeoGebra. You are sure to make some exciting discoveries. And be sure to write them up!

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