The Constants of Mathematics – Part II

The remarkable number *e*

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In this article, which is the second of our series on mathematical constants, we feature one of the most remarkable numbers in all of mathematics—the number *e*, a number that is bound to occupy centre place in any account of mathematics (along with its close cousin, the number π). It turns out that there is so much to say about *e* that we will need to devote two articles to this constant alone!

Probably the question that occurs to a reader who is meeting *e* for the first time is, why use the letter *e* for this number? Historically, it was the mathematician Leonhard Euler who first drew attention to the number and he gave it the symbol *e*. He probably chose this letter for its association with exponential functions and he may have used *e* to denote "***exponential number***." Today, we may like to think of it as "***Euler's number***." However, Euler was not the first mathematician to bump into the number; others had done so before him, but he was the first to realise its significance in mathematics. The web page [2] describes this beautifully:

> *The number e first comes into mathematics in a very minor way. This was in 1618 when, in an appendix to Napier's work on logarithms, a table appeared giving the natural logarithms of various numbers. However, that these were logarithms to base e was not recognised since the base to which logarithms are computed did not arise in the way that logarithms were thought about at this time. Although we now think of logarithms as the exponents to which one must raise the base to get the required number, this is a modern way of thinking. We will come back to this point later in this essay. This table in the appendix, although carrying no author's name, was almost certainly written by Oughtred. A few years later, in 1624, again e almost made it into the mathematical literature, but not quite. In that year Briggs gave a numerical approximation to the base 10 logarithm of e but did not mention e itself in his work.*

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As can be seen from this quote, mathematicians had been holding e in their hands for many decades without realising it! The full story of the discovery (or should we say, the invention) of logarithms by John Napier makes for a fascinating study, but we shall not go into this for now. That requires a separate article all to itself!

After a series of hits and misses, it was in 1683 that the number e was first discovered. In that year, Jakob Bernoulli (one of the early members of the remarkable Bernoulli clan), while studying compound interest, considered the possibility of 'continuous compounding' and naturally encountered the limit of $\left(1+\frac{1}{n}\right)^n$ as n tends to infinity. He was able to show, using the binomial theorem, that the limit lies between 2 and 3. But it seems that he did not pursue this line of thinking beyond this point. So though we can say that e had finally been discovered, this is more a matter of historical hindsight; Bernoulli himself did not realise the significance of what he had discovered. In particular, he did not assign any name to the constant that he had discovered.

The big year when e finally made its appearance under the name we use today is 1731, in a letter that the mathematician Leonhard Euler wrote to Christian Goldbach. He mentioned the same limit that Bernoulli had given, showed that e is a sum of an infinite series (one with which we are very familiar these days), and calculated e to 18 decimal places. But he went well beyond this point; in this early work, he defined and explored the exponential function (defined not just over the real numbers but over the complex numbers as well) and brought out the connection between the exponential function and the sine and the cosine functions.

So though there can be dispute about the precise year of birth of e, there is no question that the year in which e becomes a true citizen of mathematics is 1731!

In the next few sections, we present a series of highlights of this remarkable number, showcasing its occurrence in numerous areas of mathematics.

Continuous compounding, à la Bernoulli.

The following formula is well-known to us: if a unit sum of money earns interest at the rate of $r\%$ per year, compounded n times a year at equal intervals (here n is a positive integer), then the amount A to which it grows at the end of one year is given by

$$A=\left(1+\frac{r/100}{n}\right)^n.$$

For the sake of exploring the underlying mathematics, we take $r=100$. The function to be explored then becomes:

$$A(n)=\left(1+\frac{1}{n}\right)^n. \tag{1}$$

We have now written $A(n)$ rather than just A, as the quantity depends on n. This is the function that Jakob Bernoulli had investigated. He noted that $A(n)$ increases with n, but the rate of increase comes down sharply with increasing n and $A(n)$ appears to reach a limiting value. Figure 1 illustrates this.

The graph of $A(n)$ indicates that as n increases without bound, the quantity $\left(1+\frac{1}{n}\right)^n$ tends to a limit which lies between 2.5 and 3. Convergence does not take place as rapidly as one may expect, as the following table illustrates:

n	10	25	50	100	1000	10000
$A(n)$	2.5937	2.6658	2.6915	2.7048	2.7168	2.7181

Even for $n=10000$, we have achieved an accuracy of only three decimal places.

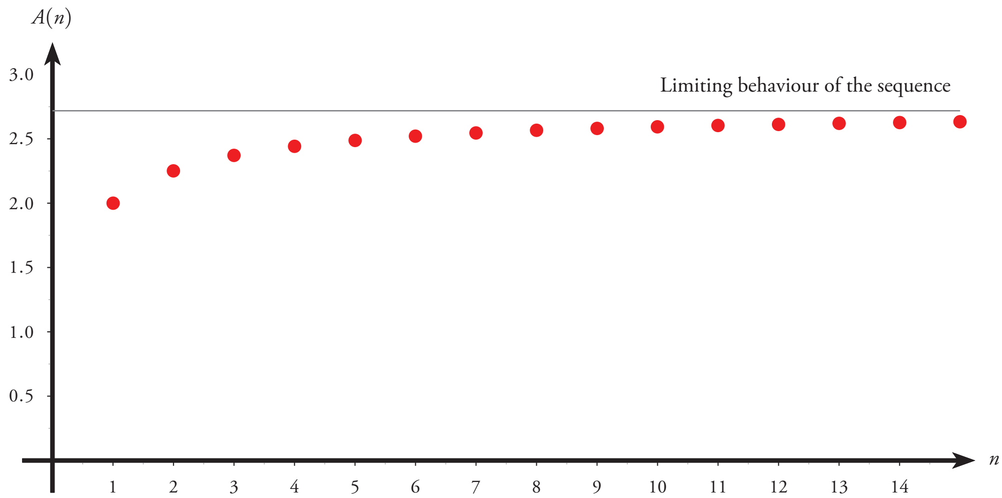

Figure 1.

To prove rigorously that $\left(1+\frac{1}{n}\right)^n$ tends to a limit takes a few steps. Generally, the proof takes the following form:

Establish that the sequence $\{A(n)\}_{n\geq 1}$ is *monotonic increasing*, i.e.,

$$A(1) < A(2) < A(3) < A(4) < \cdots.$$

In other words, we need to prove the following:

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1} \quad \text{for all } n \in \mathbb{N}. \tag{2}$$

Proving this inequality presents a nice challenge. Here is one way of proving it. For any positive integer n, we consider the following $n+1$ positive quantities:

$$\underbrace{1+\frac{1}{n},\ 1+\frac{1}{n},\ 1+\frac{1}{n},\ 1+\frac{1}{n},\ \ldots,\ 1+\frac{1}{n}}_{n \text{ of these quantities}},\ 1.$$

Their geometric mean is

$$\left(1+\frac{1}{n}\right)^{n/(n+1)}.$$

The sum of the quantities is equal to

$$n\left(1+\frac{1}{n}\right)+1 = n+2,$$

hence their arithmetic mean is equal to

$$\frac{n+2}{n+1} = 1+\frac{1}{n+1}.$$

Invoking the result that the geometric mean of a list of positive numbers which are not all equal to one another is always strictly less than the arithmetic mean of the same list of numbers (this is the celebrated "AM-GM inequality"), we get

$$\left(1+\frac{1}{n}\right)^{n/(n+1)} < 1+\frac{1}{n+1},$$

i.e.,

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}.$$

Establish that the sequence $\{A(n)\}_{n\geq 1}$ is *bounded above*. In this particular case, we only need to establish that $A(n) < 3$ for all $n \in \mathbb{N}$. One way of proving this is to use the binomial theorem. First we verify computationally that $A(3) < 3$. Next we have, for all $n \geq 4$,

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= 1+n\cdot\frac{1}{n}+\frac{n(n-1)}{2!}\cdot\frac{1}{n^2}+\frac{n(n-1)(n-2)}{3!}\cdot\frac{1}{n^3}+\cdots+\frac{1}{n^n}\\ &< 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}\\ &< 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n} \qquad (3)\\ &< 3. \qquad (4)\end{aligned}$$

A line of explanation is needed for the last two results. Inequality (3) follows from the claim

$$n! > 2^n \quad \text{for all } n \in \mathbb{N},\ n \geq 4,$$

which is best proved using induction (starting with $4! > 2^4$ as the base or 'anchor' of the induction), but we leave the details to the reader.

In the case of inequality (4), we simply use the well-known fact that

$$\sum_{n=0}^{\infty}\frac{1}{2^n} = 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots = 2.$$

With these two results established, it remains only to invoke a standard result from analysis ("a monotonically increasing sequence which is bounded above possesses a limit") which ensures that the sequence under study possesses a limit. We call the limit e. So the following may be considered to be a *definition* of e:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n. \qquad (5)$$

The following is now a straightforward consequence of (5):

$$e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots = \sum_{n=0}^{\infty}\frac{1}{n!}. \qquad (6)$$

It is interesting to check the rate of convergence of the above infinite series for e. Let $B(n)$ denote the sum $1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}$; then we have the following data:

n	10	25	50	100
$B(n)$	1.718281801	1.718281828	1.718281828	1.718281828

We see that the rate of convergence is quite rapid: $B(25)$, $B(50)$ and $B(100)$ agree to all the decimal places shown. Here is the value of e to 50 decimal places:

$$e = 2.71828\ 18284\ 59045\ 23536\ 02874\ 71352\ 66249\ 77572\ 47093\ 69995\ 9\ldots.$$

Rational approximations for e. The double occurrence of the string '1828' in the above decimal expansion can prove misleading; a calculation to 9 decimal places might suggest that the value of e is the recurring decimal 2.7 1828 1828 1828 .... This is, of course, not so. However, this line of thinking yields the following rational approximation for e:

$$\frac{271801}{99990}.$$

The difference between this and e is roughly 2.8×10^{-10}. An approximation using smaller numbers is $\frac{27180}{9999}$; this differs from e by roughly 10^{-5}. But this is by no means the best possible rational approximation for e using relatively smaller numbers. A very much better approximation is

$$\frac{23225}{8544},$$

which differs from e by roughly 6.7×10^{-9}.

You may wonder how we hit upon the approximation $\frac{23225}{8544}$. The answer is that it comes from the simple continued fraction for e. However, we leave that discussion to the second part of this article.

Another limit for e. Only a couple of decades back, it was noticed that e can be expressed as a limit in another way which converges more rapidly than the expression used earlier. Namely, we have:

$$e = \lim_{n\to\infty} \left(\frac{(n+1)^{n+1}}{n^n} - \frac{n^n}{(n-1)^{n-1}} \right). \qquad (7)$$

Writing $g(n)$ for the expression on the right side of (7), we have the following data:

n	10	50	100	500
$g(n)$	2.7194	2.718327	2.718293	2.7182823

It is not at all difficult to see intuitively why (7) is true. We urge you to find an intuitive justification for yourself. However, to devise an analytically rigorous proof takes more effort. We shall not go into the details in this article.

When the slope is 1

One of the many remarkable aspects of the number e is that it can be defined in several different ways, and these different definitions turn out to be all equivalent to each other. Here is one such way, in which we use the notion of slope of a curve.

We consider curves of the form $y = a^x$, where $a > 1$ is any real number. Figure 2 shows a few such curves. Since $a^0 = 1$ for all such a, all these curves pass through the point $P(0, 1)$. It is clear that the slope of the curve at P depends on a; let this slope be denoted by $h(a)$. The larger the value of a, the greater will be the slope. For $a = 1$, this slope is 0 (trivially so), and the slope can be made as large as we may please by taking a to be large enough.

It seems reasonable to suppose that there is a critical value of a such that $h(a) = 1$; i.e., such that the slope of the curve $y = a^x$ at the point $P(0, 1)$ is equal to 1. (This supposition can be justified rigorously, using standard methods of analysis, but we shall not go into the details here.) ***We define this critical number to be e.***

It is relatively easy to show that this definition leads to the same limit definition that we had adopted earlier. To show how, let n be any large positive integer, and consider the point Q on the curve $y = e^x$

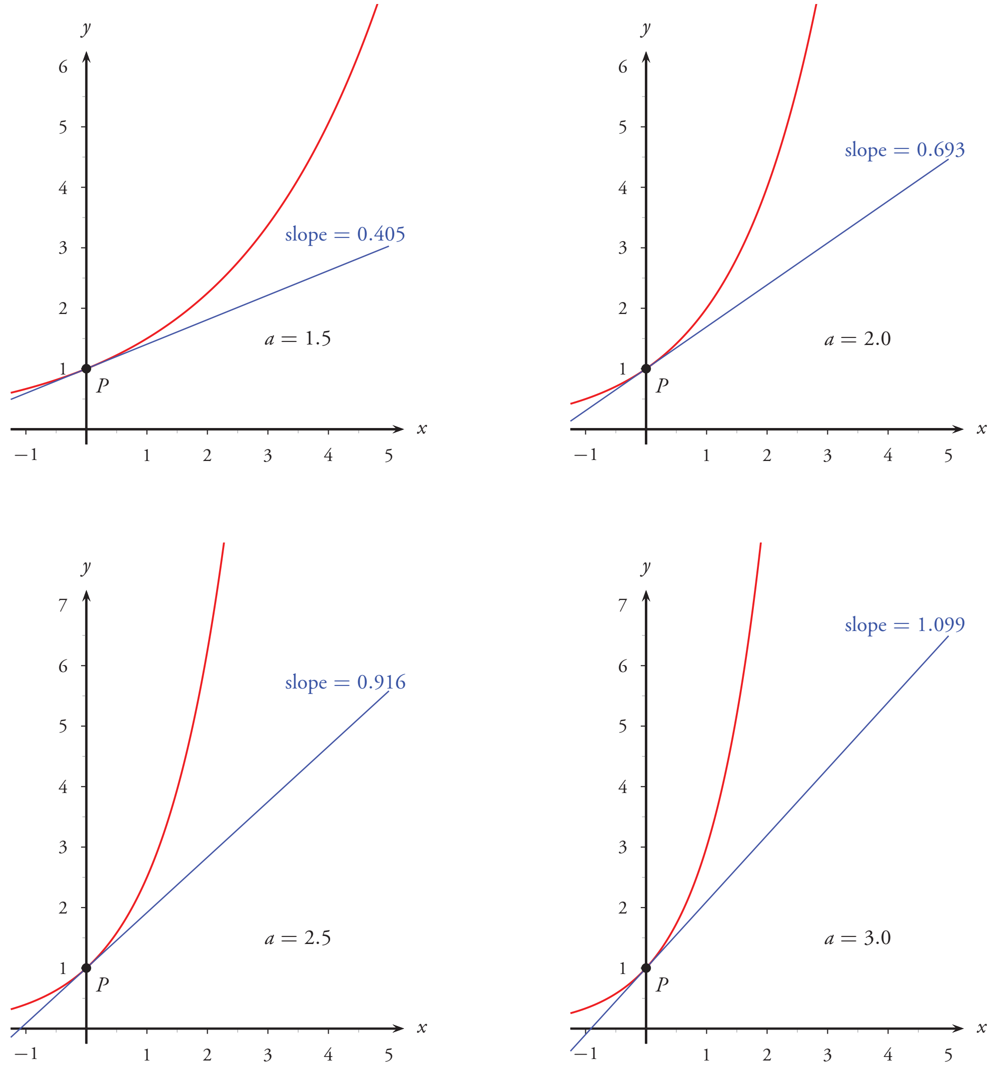

Figure 2. The curve $y = a^x$ for different values of a

(with e as just defined), with x-coordinate equal to $\frac{1}{n}$. That is,

$$Q = \left(\frac{1}{n},\ e^{1/n}\right).$$

The slope of chord PQ is equal to

$$\frac{e^{1/n} - 1}{1/n - 0} = n\left(e^{1/n} - 1\right).$$

If we let $n \to \infty$, then in the limit, the slope of chord PQ tends to the slope of the curve $y = e^x$ at the point P. By definition of the value of e, this slope is equal to 1. It follows that for large values of n, we must have

$$n\left(e^{1/n} - 1\right) \approx 1, \quad \therefore\ e^{1/n} - 1 \approx \frac{1}{n}, \quad \therefore\ e^{1/n} \approx 1 + \frac{1}{n}.$$

This implies that for large values of n, we must have

$$e \approx \left(1 + \frac{1}{n}\right)^n.$$

From this, it only takes a moment to conclude that

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

When the area is 1

Remarkably, there is also an approach using areas to hit upon the number e. This time, we consider the curve $y = \frac{1}{x}$. Figure 3 shows a sketch of the curve in the first quadrant. Let us define a function $f(t)$ for positive numbers t as follows: $f(t)$ = the area enclosed by the curve, the x-axis and the lines $x = 1$ and $x = t$. More precisely, it is the area of region $PQRS$ when it is traversed in an anticlockwise direction (see Figure 3 for the definitions of these points). Then f is a continuous function, and $f(1) = 0$.

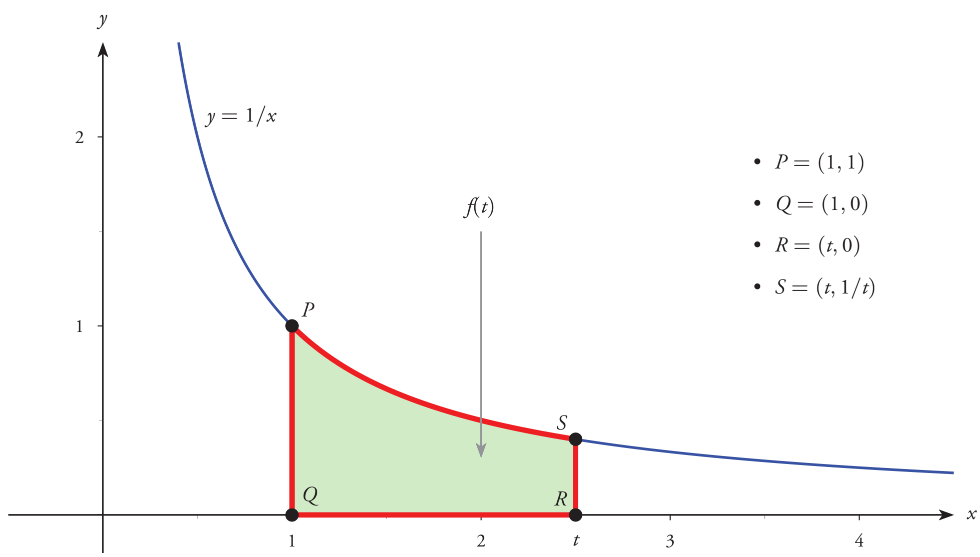

Figure 3.

The region $PQRS$ does not belong to any of the classes of regions studied up to class 10, as one of its sides is a curve which is not an arc of a circle. So it may not be obvious how we can find its area. (Remember that as far as this article is concerned, we have not yet started studying integration!) We therefore use a method of approximation, by inscribing rectangles within the region and by circumscribing rectangles about the region. By making these rectangles narrower and narrower, we get increasingly better approximations to the desired area.

For example, consider $f(2)$. In Figure 4 (a), we have drawn a rectangle inside $PQRS$. Its area is $1 \times 0.5 = 0.5$; hence $f(2) > 0.5$. In Figure 4 (b), we have drawn a trapezoid circumscribing $PQRS$. Its area is $\frac{1}{2}(1 + 0.5) \times 1 = 0.75$; hence $f(2) < 0.75$. It follows that $0.5 < f(2) < 0.75$. By drawing rectangles and trapezoids of width 0.5 (inscribed within the region and circumscribed about the region, respectively), we find that $0.58 < f(2) < 0.71$. These bounds show that $f(2) < 1$.

We may similarly consider $f(3)$. By inscribing narrower and narrower rectangles within the region and circumscribing narrower and narrower trapezoids about the region, we find that $1.04 < f(3) < 1.1$. These bounds show that $f(3) > 1$.

So we observe that $f(2) < 1$ and $f(3) > 1$. By continuity, we expect that there exists a critical value of t lying between 2 and 3 such that $f(t) = 1$. ***This critical value is called e.***

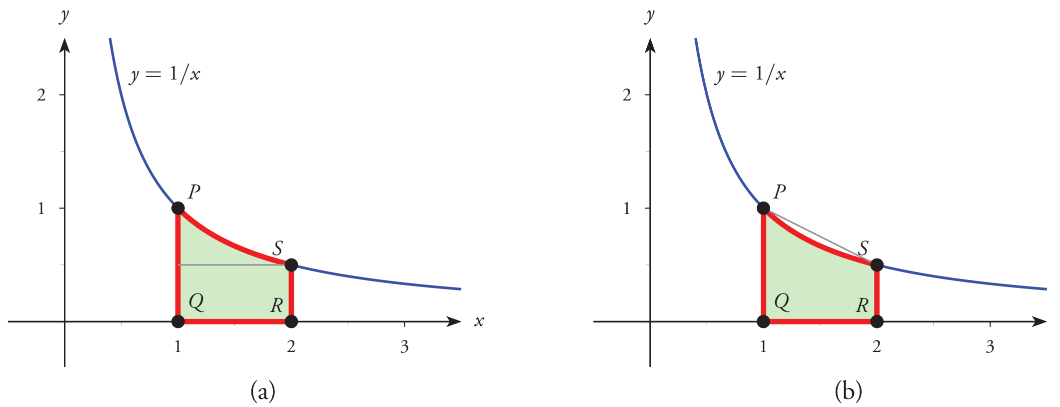

Figure 4.

Having defined e this way, the onus is now on us to show that this definition implies that

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

Perhaps the reader could look for a proof of this claim on his or her own. However, as this article has already become somewhat long, and the proof is clearly not one that will fit in the margin of this page, we shall omit the proof for now and give it in the next part of this article, which will appear in the March 2019 issue of *At Right Angles*.

A mnemonic for the exponential number

A popular aspect of mathematical culture is to find easy-to-remember mnemonics for the decimal expansions of well-known numbers such as π and e. In [1], a mnemonic is presented for the first 40 digits of e. Since 0 presents an obvious difficulty in the design of such a mnemonic, the author uses an exclamation mark to represent 0. Commas, colons, semi-colons, quote signs and full-stop signs are to be ignored, and the position of the decimal point is assumed to be known. Here is his mnemonic:

> We present a mnemonic to memorize a constant so exciting that Euler exclaimed '!' when first it was found, yes, loudly '!.' My students perhaps will compute e, use power or Taylor series, an easy summation formula, obvious, clear, elegant!

Counting out the letters, we get the digits of e: 2.718281828 . . . .

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