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A Note on Armstrong Numbers

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In this article, student Satvik Kaushik investigates Armstrong numbers. Mathematical investigation is a powerful way for students to learn more about concepts that interest them. In mathematical investigations, students are expected to pose their own problems after initial exploration of the mathematical situation. The exploration of the situation, the formulation of problems and their solution give opportunity for the development of independent mathematical thinking and in engaging in mathematical processes such as organizing and recording data, pattern searching, conjecturing, inferring, justifying and explaining conjectures and generalizations. It is these thinking processes which enable an individual to learn more mathematics, apply mathematics in other disciplines and in everyday situations and to solve mathematical (and non-mathematical) problems.

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In this note we provide a way for identifying Armstrong numbers. We also discuss their generalizations.

Introduction

In recreational number theory, an Armstrong number (named after Michael F. Armstrong) of the first kind is a number that is the sum of its own digits each raised to the power of the number of digits. For example, $153 = 1^3 + 5^3 + 3^3$. In this note we study three-digit Armstrong numbers of the first kind and their patterns in the general case (Armstrong numbers of the second kind) and we also list all the n-digit Armstrong numbers of the first kind.

Theorem 1. **There are only four three-digit Armstrong numbers of the first kind. They are 153, 370, 371 and 407.**

Proof. Consider a three-digit number $N = 100A + 10B + C$ where $A, B, C \in \{0, 1, 2, \ldots .9\}$ and $A \neq 0$. Assuming that it is an Armstrong number, we shall find the possible values of A, B, C.

From the definition of an Armstrong number, it follows that $A^3 + B^3 + C^3 = 100A + 10B + C$. We consider the different possible values of A.

Suppose that $A = 1$. Then $B^3 + C^3 = 10B + C + 99$. The number on the right side is at least 99 and at most 198. Hence $B < 6$ and $C < 6$. We can check that either $B = C = 4$, or one of B or C is 5. (The other possibilities clearly do not work out.)

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Clearly $(B, C) = (4, 4)$ does not satisfy $B^3 + C^3 = 10B + C + 99$.

Hence one of B or C is 5. If $B = 5$, then $C^3 - C = 24$, so $C = 3$, hence the number is 153.

If $C = 5$, then $B^3 + 125 = 10B + 104$, so $B^3 - 10B = -21$, but this does not yield any positive integer solution for B.

Hence the first three-digit Armstrong number is 153.

That is, $\mathbf{1^3 + 5^3 + 3^3 = 1 + 125 + 27 = 153.}$

Next, suppose that $A = 2$. Then $B^3 + C^3 = 10B + C + 192$. The number on the right side is at least 192 and at most 291, hence either $B = C = 5$, or one of B or C is 6.

In neither case do we find any such B and C which fits the equation. So there is no three-digit Armstrong number starting with 2.

Next, suppose that $A = 3$; then $B^3 + C^3 = 10B + C + 273$. The number on the right side is at least 273 and at most 372, which implies $(B, C) = (5, 6)$ or $(6, 5)$ or B or C is 7. It is easy to check that the possibilities $(B, C) = (5, 6)$ and $(B, C) = (6, 5)$ do not work. Now consider the possibility $C = 7$; in this case $B^3 = 10B - 63$, but this gives no such B. Next, consider $B = 7$; in this case we have $C^3 = C$, which gives $C = 0$ or $C = 1$. Hence the possible numbers are $370, 371$. Equality works out for both these numbers.

That is, $\mathbf{3^3 + 7^3 + 0^3 = 27 + 343 + 0 = 370}$ and $\mathbf{3^3 + 7^3 + 1^3 = 27 + 343 + 1 = 371.}$

Now let us proceed with the case $A = 4$. In this case we have $B^3 + C^3 = 10B + C + 336$. The number on the right side is at least 336 and at most 435, which implies $(B, C) = (5, 6)$ or $(6, 5)$ or B or C is 7. It is easy to check that no solution exists with either $B = 7$ or $(B, C) = (5, 6)$ or $(6, 5)$. Hence we take $C = 7$, which gives $B^3 = 10B$. This yields $B = 0$. Hence the number is 407. This fits the requirement.

That is, $\mathbf{4^3 + 0^3 + 7^3 = 64 + 0 + 343 = 407.}$

If we experiment with the possibilities $A \in \{5, 6, 7, \ldots, 9\}$, we do not find any solutions for B and C.

Hence there are only four three-digit Armstrong numbers of the first kind; they are **153, 370, 371, 407.**

Note. In [1], the author Dr. MOLOY DE gives a brief description on Armstrong numbers as follows:

"Armstrong numbers of first kind are base dependent and they are certainly rare. They cannot have more than 60 digits in base 10, because for $n > 60$, $n9^n < 10^{n-1}$. Since there is an upper limit to their size, it is theoretically possible to find all of them, given sufficient computer time. However, 10^{60} is an unimaginably huge number, so such a 'brute force' approach would be unwise. Luckily, D. Winter proved in 1985 that there are exactly 88 base-10 Armstrong numbers of first kind, and they must have 1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 14, 16, 17, 19, 20, 21, 23, 24, 25, 27, 29, 31, 32, 33, 34, 35, 37, 38 or 39 digits. Of course, the one-digit Armstrong numbers of first kind are somewhat trivial since clearly $1^1 = 1, 2^1 = 2$ etc. The Armstrong numbers of first kind up to 10 digits are 1, 2, 3, 4, 5, 6, 7, 8, 9, 153, 370, 371, 407, 1634, 8208, 9474, 54748, 92727, 93084, 548834, 1741725, 4210818, 9800817, 9926315, 24678050, 24678051, 88593477, 146511208, 472335975, 534494836, 912985153, and 4679307774. The largest Armstrong number of first kind (in base 10) is the 39-digit beast: 115132219018763992565095597973971522401".

The following table gives a list of all n-digit Armstrong numbers of the first kind [2], [3].

No.	Armstrong Numbers of first kind
1	0, 1, 2, 3, 4, 5, 6, 7, 8, 9
3	153, 370, 371, 407
4	1634, 8208, 9474
5	54748, 92727, 93084
6	548834
7	1741725, 4210818, 9800817, 9926315
8	24678050, 24678051, 88593477
9	146511208, 472335975, 534494836, 912985153
10	4679307774, 32164049650, 32164049651, 40028394225, 42678290603, 44708635679, 49388550606, 82693916578,
11	94204591914
14	28116440335967
16	4338281769391370, 4338281769391371
17	21897142587612075, 35641594208964132, 35875699062250035
19	1517841543307505039, 3289582984443187032, 4498128791164624869, 4929273885928088826
20	63105425988599693916
21	128468643043731391252, 449177399146038697307
23	21887696841122916288858, 27879694893054074471405, 27907865009977052567814, 28361281321319229463398, 35452590104031691935943
24	174088005938065293023722, 188451485447897896036875, 239313664430041569350093
25	1550475334214501539088894, 1553242162893771850669378, 3706907995955475988644380, 3706907995955475988644381, 4422095118095899619457938
27	121204998563613372405438066, 121270696006801314328439376, 128851796696487777842012787, 174650464499531377631639254, 177265453171792792366489765
29	14607640612971980372614873089, 19008174136254279995012734740, 19008174136254279995012734741, 23866716435523975980390369295
31	1145037275765491025924292050346, 1927890457142960697580636236639, 2309092682616190307509695338915
32	17333509997782249308725103962772
33	186709961001538790100634132976990, 186709961001538790100634132976991
34	1122763285329372541592822900204593
35	12639369517103790328947807201478392, 12679937780272278566303885594196922
37	1219167219625434121569735803609966019
38	12815792078366059955099770545296129367
39	115132219018763992565095597973971522400, 115132219018763992565095597973971522401

Generalisations of Armstrong numbers

Now we discuss some generalizations of Armstrong numbers. In Theorem 1 we discussed n-digit Armstrong numbers of the first kind. Now we discuss the Armstrong numbers of second kind. Such a number has the property that it "is equal to the sum of the cubes of numbers composed of two successive digits or three successive digits or four successive digits and so on of the number." For example:

$153 = 1^3 + 5^3 + 3^3$ (Armstrong number of first kind);

$165033 = 16^3 + 50^3 + 33^3$ (Armstrong number of second kind);

$166500333 = 166^3 + 500^3 + 333^3$ (Armstrong number of second kind);

$166650003333 = 1666^3 + 5000^3 + 3333^3$ (Armstrong number of second kind).

First we prove the example stated above, then we explore these numbers and their generalizations.

Theorem 2.

$$\mathbf{1666\ldots^3 + 5000\ldots^3 + 3333\ldots^3 = 1666\ldots6\,5000\ldots0\,3333\ldots3},$$

where the numbers of 6's, 3's and 0's are the same in the three numbers on the left side.

Proof. Here $1^3 + 5^3 + 3^3 = 1 + 125 + 27 = 153$, $16^3 + 50^3 + 33^3 = 165033$. Here we can see a pattern among the results. To prove the pattern for all A, B, C, take A, B, C in terms of variable "n" such that we can say

n	A	B	C	ABC
1	$1 = \frac{10^1 - 4}{6}$	$5 = \frac{10^1}{2}$	$3 = \frac{10^1 - 1}{3}$	$153 = 1(10^2) + 5(10^1) + 3(10^0)$ $= \left(\frac{10^1 - 4}{6}\right)(10^{2(1)}) + \left(\frac{10^1}{2}\right)(10^{1(1)}) + \left(\frac{10^1 - 1}{3}\right)(10^{0(1)})$
2	$16 = \frac{10^2 - 4}{6}$	$50 = \frac{10^2}{2}$	$33 = \frac{10^2 - 1}{3}$	$165033 = 16(10^4) + 50(10^2) + 33(10^0)$ $= \left(\frac{10^2 - 4}{6}\right)(10^{2(2)}) + \left(\frac{10^2}{2}\right)(10^{1(2)}) + \left(\frac{10^2 - 1}{3}\right)(10^{0(2)})$
3	$166 = \frac{10^3 - 4}{6}$	$500 = \frac{10^3}{2}$	$333 = \frac{10^3 - 1}{3}$	$166500333 = 166(10^6) + 500(10^3) + 333(10^0) = \left(\frac{10^3 - 4}{6}\right)(10^{2(3)}) + \left(\frac{10^3}{2}\right)(10^{1(3)}) + \left(\frac{10^3 - 1}{3}\right)(10^{0(3)})$
4	$1666 = \frac{10^4 - 4}{6}$	$5000 = \frac{10^4}{2}$	$3333 = \frac{10^4 - 1}{3}$	$166650003333 = 1666(10^8) + 5000(10^4) + 3333(10^0) = \left(\frac{10^4 - 4}{6}\right)(10^{2(4)}) + \left(\frac{10^4}{2}\right)(10^{1(4)}) + \left(\frac{10^4 - 1}{3}\right)(10^{0(4)})$

$A = \dfrac{10^n - 4}{6}$, $B = \dfrac{10^n}{2}$ and $C = \dfrac{10^n - 1}{3}$.

Hence

$$\text{LHS} = A^3 + B^3 + C^3 = \left(\frac{10^n - 4}{6}\right)^3 + \left(\frac{10^n}{2}\right)^3 + \left(\frac{10^n - 1}{3}\right)^3 = \frac{10^{3n} - 10^{2n} + 2.10^n - 2.10^0}{6},$$

$$\text{RHS} = ABC = \left(\frac{10^n - 4}{6}\right)(10^{2n}) + \left(\frac{10^n}{2}\right)(10^n) + \left(\frac{10^n - 1}{3}\right)(10^0) = \frac{10^{3n} - 10^{2n} + 2.10^n - 2.10^0}{6}.$$

So $A^3 + B^3 + C^3 = 100A + 10B + C$ where $A = \dfrac{10^n - 4}{6}$, $B = \dfrac{10^n}{2}$ and $C = \dfrac{10^n - 1}{3}$.

Hence

$$\mathbf{1666\ldots}^3 + \mathbf{5000\ldots}^3 + \mathbf{3333\ldots}^3 = \mathbf{1666\ldots6\ 5000\ldots0\ 3333\ldots3}.$$

Theorem 3. **There exist Armstrong numbers of the second kind corresponding to all three-digit Armstrong numbers of the first kind.**

Proof. Let us start with an Armstrong number of the first kind, say $100A + 10B + C = 370$.

Take the six-digit number $\overline{A_0B_0C_0}$ where A_0, B_0, C_0 are two-digit numbers such that

$$A_0^3 + B_0^3 + C_0^3 = 10^4 A_0 + 10^2 B_0 + C_0,$$

where A_0 has the form $\overline{3a}$ or $\overline{a3}$, B_0 has the form $\overline{7b}$ or $\overline{b7}$, and C_0 has the form $\overline{0c}$ or $\overline{c0}$.

Let us start with the possibility $A_0 = \overline{3a}$, that is, $A_0^3 + B_0^3 + C_0^3$ is a six-digit number starting with 3 and followed by a. There are 4 possible cases to consider.

Let us take the case where $B_0 = \overline{7b}$ and $C_0 = \overline{0c}$.

It should be clear that $30^3 \le \left(\overline{3a}\right)^3 \le 39^3$. We get similar bounds for B_0^3 and C_0^3.

Now we make use of the following relations:

- minimum possible value of $B_0^3 + C_0^3$ is equal to minimum possible value of $\left(A_0^3 + B_0^3 + C_0^3\right)$ − maximum possible value of A_0^3;
- maximum possible value of $B_0^3 + C_0^3$ is equal to maximum possible value of $\left(A_0^3 + B_0^3 + C_0^3\right)$ − minimum possible value of A_0^3.

From this we deduce that $B_0^3 + C_0^3$ lies between 240681 and 372999.

Since C_0^3 does not exceed 729 and $72^3 > 372999$, B_0 can be only 70 or 71.

If $B_0 = 70$, then $A_0^3 + C_0^3 \le 56999$. Hence A_0 lies between 30 and 38.

Here first two digits are 34, hence $A_0 \in [34, 38]$.

Hence $A_0^3 + B_0^3 + C_0^3 = (3a)^3 + 343000 + c^3$. It is clear that no 'a' satisfies this equation. Similarly, B_0=71 also doesn't give any solution.

Now let us take the case where $B_0 = \overline{b7}$ and $C_0 = \overline{0c}$.

We know that $0 \le C_0^3 \le 729$. It follows that B_0^3 lies between 239952 and 372999, hence $B_0 = 67$. This yields $\left(\overline{3a}\right)^3 + 300763 + \left(\overline{0c}\right)^3 = \overline{3a670c}$; from this we get $a = 3$ and $c = 0$ or 1.

If we continue the same for $B_0 = \overline{7b}$, $C_0 = \overline{c0}$, we get no solution.

If we continue the same for $B_0 = \overline{b7}$, $C_0 = \overline{0c}$, we get $B_0 = 67$ and $C_0 = 0$ or 1.

That is, if $A_0 = \overline{a3}$, then we get $A_0 = 33$, $B_0 = 67$, $C_0 = 00$ or 01.

Now observe that $3^3 + 7^3 + 0^3 = 370$ and $33^3 + 67^3 + 00^3 = 336700$.

This seems to be in the form $A^3 + B^3 + C^3 = \overline{ABC}$ where $A = \dfrac{10^n - 1}{3}$, $B = \dfrac{2.10^n + 1}{3}$ and $C = 000\ldots$ (n times). Since

$$A^3 + B^3 + C^3 = \left(\frac{10^n - 1}{3}\right)^3 + \left(\frac{2.10^n + 1}{3}\right)^3 + 0^3 = \frac{10^{3n} + 10^{2n} + 10^n}{3},$$

$$ABC = \left(\frac{10^n - 1}{3}\right)(10^{2n}) + \left(\frac{2.10^n + 1}{3}\right)(10^n) + 0(10^0) = \frac{10^{3n} + 10^{2n} + 10^n}{3},$$

it is true.

Hence the generalization for 153:

$$\mathbf{1666\ldots^3 + 5000\ldots^3 + 3333\ldots^3 = 1666\ldots6\ 5000\ldots0\ 3333\ldots3}.$$

The same generalisation works for the numbers 370, 371 and 407. These yield the four Armstrong numbers of the second kind.

Hence there exist Armstrong numbers of the second kind for all three-digit Armstrong numbers of the first kind.

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Reference

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