ClassRoom

TearOut

Fun with Dot Sheets

Beginning with the last issue, we started the TearOut series. In this article, we focus on investigations with graph paper. Pages 3 & 4 give guidelines for the facilitator, pages 1 & 2 are a worksheet for students. This time we explore quadrilaterals and triangles using lattice points.

SWATI SIRCAR

The following activities can be done on centimeter or inch graph paper, rectangular dot sheets or square grids. They start with the first quadrant where all coordinates are non-negative. The origin is kept as one of the starting points so that patterns can be seen easily. Later, the activities have been generalized to other quadrants and starting points.

Geometry on graph paper

Get some graph papers, a pencil and a scale… you are all set to go… ☺

Quadrilaterals

1. **Take the origin O and any 2 points P and Q in the 1st quadrant, draw OP and OQ and complete the parallelogram OPSQ**
    - 1.1. How are the coordinates of S related to those of P and Q?
    - 1.2. How are the coordinates of the intersection of OS and PQ related to those of S?
    - 1.3. When will OPSQ be (i) a rectangle? (ii) a rhombus? (iii) a square?
    - 1.4. When will O, P and Q not form three vertices of a parallelogram? In that case, can they be three vertices of any other quadrilateral? How?
2. **Take origin O and any point P in the 1st quadrant, draw OP and**
    - 2.1. Draw a square OPSQ
        - 2.1.1. What are the possible coordinates of Q? How are they related to the coordinates of P?
        - 2.1.2. What are the coordinates of S? How are they related to those of P and Q?
    - 2.2. Draw a rhombus OPSQ and explore 2.1.1 and 2.1.2
    - 2.3. Draw a rectangle OPSQ and explore 2.1.1 and 2.1.2
3. *[optional] Take origin O and any point P in any quadrant and repeat all of 2.*
4. **Take origin O and any 2 points P and Q. Draw OP and OQ. Draw the parallelogram OPS_1Q. Draw PQ and the parallelograms $OPQS_2$ and OS_3PQ**
    - 4.1. What are the coordinates of S_1, S_2 and S_3?
    - 4.2. How are they related to the coordinates of P and Q?
    - 4.3. Comment on ΔOPQ and $\Delta S_1S_2S_3$
5. **Take any 3 points T, P and Q, draw the parallelograms TPS_1Q, $TPQS_2$, TS_3PQ and repeat 4.**

Triangles

1. **Take origin O and any point P on the positive x-axis and draw a scalene ΔOPQ such that Q is in the 1st quadrant (See Figure 1)**
   1.1. Comment on the coordinates of Q when ∠OPQ is
   (i) acute (ii) right (iii) obtuse
   1.2. If Q is on the y-axis comment on ∠QOP.

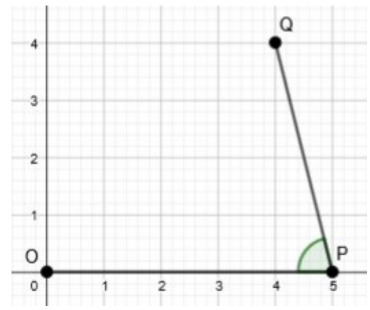

Figure 1

2. **Take origin O and any point P on the positive x-axis and draw an isosceles ΔOPQ such that Q is in the 1st quadrant and OQ = PQ (See Figure 2)**
   2.1. How are the coordinates of Q related to those of P?
   2.2. What is the height of ΔOPQ? How is that related to the coordinates of any of the points?
   2.3. Consider the following cases:
   2.3.1. If height = ½ × base i.e. ½ OP
   2.3.2. If height < ½ × base i.e. ½ OP
   2.3.3. If height > ½ × base i.e. ½ OP
   2.3.4. If height > base i.e. OP
   Comment on ∠OQP and check which is true: (i) OP > OQ or (ii) OP < OQ for each case

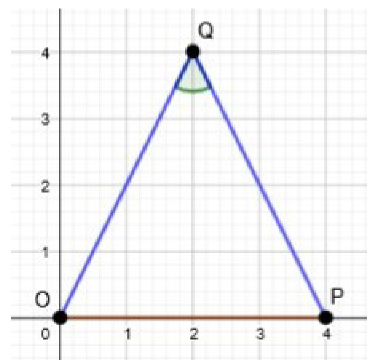

Figure 2

3. *[optional] Take origin O and any point P on the positive y-axis and repeat all of 1 and 2. (Q on x-axis in 1.2)*
4. *[optional] Take origin O and any point P on any of the axes and repeat all of 1 and 2*
5. **Take origin O and any point P in the 1st quadrant and draw a isosceles ΔOPQ such that Q is also in 1st quadrant and OP = OQ (See Figure 3)**
   5.1. How are the coordinates of Q related to those of P?
   5.2. If P is on the x-axis, where is Q? Comment on ∠POQ in that case.
   5.3. When will OP not form an isosceles ΔOPQ with OP = OQ? In that case,
   5.3.1. Can you get an isosceles triangle in the 1st quadrant with OP = PQ?
   5.3.2. How are the coordinates of Q related to those of P in that case?
   5.3.3. Comment on ∠OPQ.

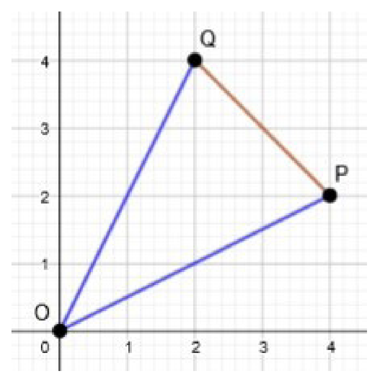

Figure 3

6. **Take origin O and any point P in any quadrant and repeat 5.1 and 5.2**
   6.1. If P is on the y-axis, where is Q? Comment on ∠OPQ in that case.
   6.2. If the x-coordinate and the y-coordinate of P are the same i.e. of the form (a, a)
   6.2.1. What are the coordinates of Q?
   6.2.2. Comment on ∠POQ in that case
7. **Take any 2 points T and P and draw triangles for all 3 angles in each case**
   7.1. a scalene ΔTPQ such that ∠TPQ (or ∠PTQ) is
   (i) acute angle (ii) right angle (iii) obtuse angle
   7.2. an isosceles ΔTPQ such that PT = PQ and ∠TPQ is
   (i) acute angle (ii) right angle (iii) obtuse angle

The only pre-requisite for these activities is familiarity with the origin, the axes – especially just the positive parts in most cases – and finding the coordinates of a given point on the plane. So this can be attempted at the upper primary level. It revisits some of the properties of parallelograms and triangles that students learn at this stage. It will also lay the foundation for some properties (or theorems) that they will learn at the secondary level.

These activities are designed to develop an intuitive sense of equal sides, right angles, etc., followed by observing the pattern in the coordinates of the points. As a starting point, we expect the students to use lattice points (i.e. points with integer coordinates) mostly. We also want students to use eye-estimate at this stage rather than construction so that they develop the intuitive sense as mentioned above. This would foster an understanding of the coordinate system and how it relates to the properties.

If done as a class, each student can record one set of values of the coordinates of P, Q and S. Later, these sets can be collated in a table. If done individually, each student will need to find different sets of values of coordinates of P, Q and S. From that tabulated data, the pattern of how these three points are related emerges and the relation can be generalized algebraically. Table 1 provides a set of values for the coordinates of P, Q, and S. Let us take a closer look activity by activity.

P	Q	S
(5, 2)	(2, 6)	(7, 8)
(3, 8)	(5, 7)	(8, 15)
(4, 2)	(7, 3)	(11, 5)
⋮	⋮	⋮

Table 1

Whenever we mention easy options, we mean lattice points. It may be possible to find other rational options and if a student finds those, then justification should be asked when needed.

Quadrilaterals

The key ideas include the vector addition (and subtraction) which is based on parallelograms (See Figure 4).

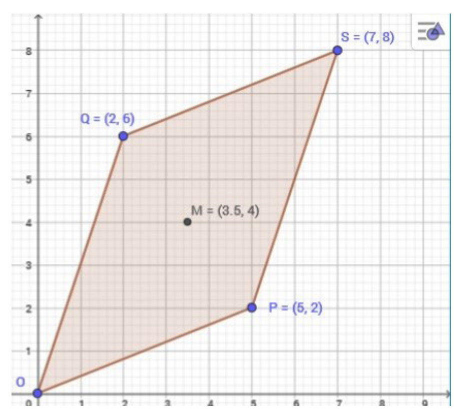

Figure 4

1. **Parallelograms**

   1.1. Vector addition: i.e. if P = (a, b) and Q = (c, d), then S = (a + c, b + d)

   1.2. Midpoint formula: Intersection of OS and PQ = $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$, in particular the coordinates are half of those of S

   1.3. Right angles ⇒ P and Q must be on the axes, equal sides ⇒ P = (a, b) and Q = (b, a) are easy options i.e.

   (i) P = (a, 0), Q = (0, b) or P = (0, a), Q = (b, 0)

   (ii) P = (a, b), Q = (b, a)

   (iii) P = (a, 0), Q = (0, a) or P = (0, a), Q = (a, 0)

   Note: If P = (a, b) where a, b are integers, it may be difficult to find another integer pair (c, d) ≠ (b, a) for Q such that OP = OQ or $a^2 + b^2 = c^2 + d^2$. One can draw circle with centre O and radius OP, but it may not go through any lattice point other than (b, a). It is possible that it may not pass through any other point that has rational coordinates.

   1.4. When O, P and Q are collinear

2. **Squares and more**

   2.1. If P = (a, b), then Q = (–b, a) or Q = (b, –a) — the coordinates are switched and one of them has the sign changed, coordinates of S = P + Q as before. The relation between coordinates of P and Q paves the way for understanding how slopes of perpendicular lines are related

2.2. Similar easy options like in 1.3 i.e. if P = (a, b) then Q = (b, a)

2.3. This is similar to finding a Q that will make ∠POQ a right angle like in 2.1, however since all sides of a rectangle are not necessarily equal there are more choices for Q like (−b, a), (−2b, 2a), (−3b, 3a) … or (b, −a), (2b, −2a), (2b, −3a) … i.e. (−nb, na) for any integer $n \neq 0$
Each one reinforces the vector addition

3. Generalizes the above with coordinates involving negative numbers

4. Creates the three possible parallelograms that are double of ΔOPQ.
$S_1 = P + Q = (a + c, b + d)$, $S_2 = Q - P = (c - a, d - b)$, $S_3 = P - Q = (a - c, b - d)$
Also O, P and Q are midpoints of S_2S_3, S_3S_1 and S_1S_2 respectively
∴ $\Delta S_1S_2S_3$ is made of four triangles congruent to ΔOPQ

5. Generalizes the vector addition and the rest of 4 away from the origin

Triangles

While construction of triangles with given specification is included in most upper primary syllabus and textbooks, students often don't develop the ability to draw or visualize different kinds of triangles on the square grid. These activities foster that intuitive sense and brings attention to the coordinates and relations among them. The formulas encountered later will become more meaningful with this experience.

1. **Scalene triangles**
   1.1. If P = (a, 0) and Q = (c, d), this fosters the understanding that ∠OPQ is greater, equal to or less than 90° if and only if $c > a$, $c = a$ and $c < a$ respectively, that ∠OPQ increases with "c" and and its being acute, right or obtuse is independent of "d"
   1.2. Reinforcing the angle between the axes

2. **Isosceles triangles**
   2.1. If P = (a, 0) then Q = (a/2, d) where $d > 0$
   2.2. That the height of ΔOPQ = d i.e. y-coordinate of Q
   2.3. Explores all possible isosceles triangles especially how the relation between the height and the base changes with the angle opposite to the base OP

   In particular, this can capture two types of acute isosceles in 2.3.3 and 2.3.4. OP > OQ for 2.3.1 i.e. right isosceles and 2.3.2 i.e. obtuse isosceles. OP < OQ for 2.3.4. The remaining case 2.3.3 is interesting since both cases are possible. It can be interesting to push students to understand what is the cut-off for OP = OQ, what kind of triangle that forms, what is the height of that triangle etc.

3. Reverses the axes for 1 and 2

4. Generalizes 1 and 2 to the negative parts of the axes

5. **More isosceles**
   5.1. Similar easy option as Quad 1.3 (ii)
   5.2. Similar to Quad 1.3 (iii)
   5.3. Similar to Quad 1.4 If P = (a, a) and connecting back to 2

6. Generalizes to coordinates with negative numbers
   6.1. If P = (0, b) then Q = (±b, 0) and ∠OPQ = 45°. Why?
   6.2. Q = (a, −a) or Q = (−a, a) i.e. reflection of P on x- or y-axis respectively forming a right angle ∠POQ

7. Generalizes away from origin, fostering the sense of drawing almost all possible scalene and isosceles triangles. It also exposes students to consider all the possibilities when side-wise and angle-wise classifications of triangles are combined. Note that one type of triangle is not included. This one can't be drawn with only lattice points as its vertices. Look out for the proof in the next issue!