LATTICE POINT GEOMETRY

Part 1

Non-existence of a Lattice-point Equilateral Triangle

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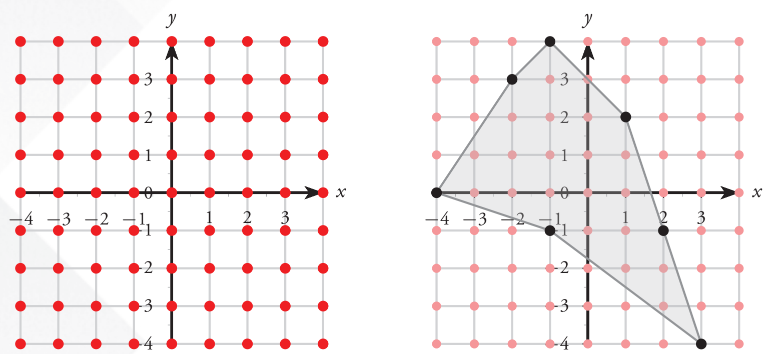
In this article, Shailesh Shirali begins with a seemingly simple question but develops the answer into not one, but four different proofs! While the content focuses on mathematics that has many applications, some of which are mentioned here, the multiplicity of proofs is an added draw, helping the teacher to illustrate innovative ways of thinking and connections across approaches.

In the cartesian plane, coordinatized by a pair of rectangular axes, we say that a given point is a ***lattice point*** if its x- and y-coordinates are both integers. For example, the points with coordinates $(0, 1)$, $(1, 2)$, $(2, -3)$ and $(-3, 7)$ are lattice points, whereas $(3, 3.5)$ and $(2.3, 1)$ are not lattice points. The set of all lattice points in the coordinate plane is called a ***lattice***. A polygon in the coordinate plane all of whose vertices are lattice points is called a ***lattice polygon***. (See Figure 1.) There are many mathematical results of great interest pertaining to lattices and to lattice polygons, and we shall talk about some of them here.



Portion of a lattice
(i)

A lattice polygon
(ii)

Figure 1

Keywords: Lattice points, lattice-point triangle, irrational number, slope, congruence, descent

The notion of a lattice actually originated in the study of crystals and is a concept derived from crystallography. The definition of a lattice that we have adopted is a slightly restricted one, and crystallographers prefer a more general definition. But we shall not venture into that area for now.

The question we ask in Part I of this multi-part article is: *Using lattice points as vertices, can we find an equilateral triangle in the plane?* In short: ***Does there exist a lattice-point equilateral triangle?*** (To avoid needless complications arising from degenerate cases, we could specify 'lattice-point equilateral triangle with nonzero area'. But we shall assume this to be the case, implicitly.)

We can ask more generally about regular polygons in the lattice plane. Trivially, there exist squares in the lattice plane. How about regular pentagons? Or regular hexagons? Or regular heptagons or octagons? Clearly, there are infinitely many questions of this kind which can be posed.

For the moment we shall focus only on the equilateral triangle. We shall see that there is some elegant mathematics involved.

There does not exist a lattice-point equilateral triangle

In this section, we show that there does not exist an equilateral triangle whose vertices are distinct lattice points. We do so in four different ways. Why so many proofs of the same result? Isn't that an overkill? Possibly—but not to this author! For one thing, all the proofs are elegant, illustrating different mathematical themes; and they all lead in different directions. Precisely because of this, the basic result may be generalised in different ways.

First proof: Argument based on area. First we note the following fact: any lattice-point $\triangle ABC$ (i.e., a triangle whose vertices are lattice points) can be enclosed within a lattice-point rectangle in such a way that each vertex of the triangle either lies on a side of the rectangle or coincides with a vertex of the rectangle. Figure 2 illustrates how this may be done. Small variations occur depending on how the triangle is oriented, but the general idea is the same. It will always happen that at least one vertex of the triangle will coincide with a vertex of the rectangle. Observe that in this configuration, the sides of the rectangle necessarily have integer lengths. The same is true for the two shorter sides ('legs') of the three right-angled triangles that surround the given lattice triangle. It follows that the area of the rectangle is an integer, and the areas of the three right-angled triangles are all half-integers. (By 'half-integer' we mean a fraction whose denominator is at most 2.) This implies that the area of the given triangle is a half-integer as well. In particular, this means that the area of $\triangle ABC$ is a rational number.

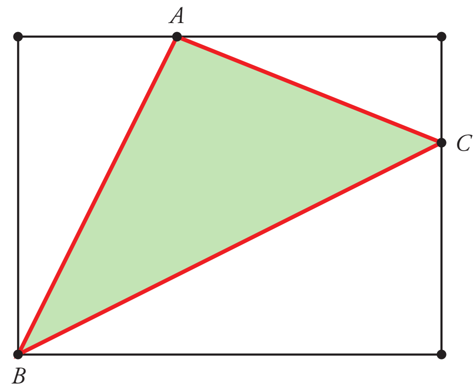


Figure 2

The argument sketched above is true for any lattice-point $\triangle ABC$. Now we consider the case when ABC is a lattice-point equilateral triangle with positive area. Let us apply the sine formula for the area of a triangle: "Area equals half the product of any two sides times the sine of the included angle." Since $\triangle ABC$ is equilateral by assumption, this yields:

$$\text{Area of } \triangle ABC = \frac{1}{2} AB^2 \cdot \sin 60^\circ = \frac{\sqrt{3}}{4} AB^2.$$

Let $A = (a, a')$ and $B = (b, b')$ where a, a', b, b' are integers. By the Pythagorean formula,

$$AB^2 = (a-b)^2 + (a'-b')^2 = \text{some nonzero integer},$$

implying that the area of the triangle is

$$\frac{\sqrt{3}}{4} \times \text{some nonzero integer}.$$

Hence the area of the triangle is an irrational number (since $\sqrt{3}$ is irrational). However, we had already concluded by a different line of argument that the area of the triangle is a rational number. So we arrive at a contradiction. That is, the assumption that there exists a lattice-point equilateral triangle leads to a contradiction and therefore cannot be true. Hence there does not exist a lattice-point equilateral triangle. □

Second proof: Argument based on slopes and angles. Our second proof is simpler than the first one and establishes a more general result. Thus, it can be said to be a stronger approach than the one used above.

The result we prove is easy to state: using only distinct lattice points as vertices, we cannot even construct a 60° angle! Clearly if this is the case, we cannot construct a lattice-point equilateral triangle either.

The proof uses the idea of slope. We claim the following: if A, B, C are three distinct lattice points such that AB is not perpendicular to BC, then $\tan \measuredangle ABC$ is a rational number. To see why, suppose that neither AB nor BC is parallel to the y-axis. Let m, n be the slopes of BA, BC respectively. Then, by a well-known formula from coordinate geometry, we have:

$$|\tan \measuredangle ABC| = \left|\frac{m-n}{1+mn}\right|.$$

By assumption, m, n are rational numbers and $mn \neq -1$. Hence $(m-n)/(1+mn)$ is a well-defined rational number; i.e., $\tan \measuredangle ABC$ is a rational number. If either AB or BC is parallel to the y-axis, then one of m, n is undefined; so we cannot use the above formula. However, the same conclusion holds. (Please fill in the details of the proof on your own.)

On the other hand, if $\measuredangle ABC = 60°$, then its tangent equals $\sqrt{3}$, an irrational number.

So the assumption that a 60° angle can be formed using only lattice points as vertices leads to a contradiction. It follows that the phenomenon is not possible at all. □

Third proof: A number-theoretic argument. Assume that there exists a non-degenerate lattice-point equilateral $\triangle ABC$. By translating the triangle parallel to itself suitably, we can make one of its vertices coincide with the origin. This gives us a lattice-point equilateral triangle in which one vertex lies at the origin of the coordinate system. Assume that this vertex is B. Let $A = (a, b)$ and $C = (c, d)$, where a, b, c, d are integers. Since the triangle is equilateral, we must have, for some positive integer k,

$$a^2 + b^2 = k,$$
$$c^2 + d^2 = k,$$
$$(a-c)^2 + (b-d)^2 = k.$$

By adding the first two equations and subtracting the third one, we obtain the following:

$$2ac + 2bd = k.$$

From this we deduce that k is even. This implies that a, b are both odd or both even, and, similarly, that c, d are both odd or both even. In short, a, b have the same parity and c, d have the same parity.

It may be the case that a, b, c, d are all even. In this case, $\triangle A'BC'$ where $A' = (a/2, b/2)$ and $C' = (c/2, d/2)$ is a lattice-point equilateral triangle as well (it has half the scale of the original triangle). The whole argument can be framed in terms of this triangle rather than the original one. By repeating this step as many times as needed, we eventually reach a stage where either A or C has at least one coordinate which is an odd number, i.e., at least one of a, b, c, d is odd. So there is no loss of generality in assuming that at least one of a, b, c, d is odd.

We now recall an important fact about square numbers: an even square is of the form 0 (mod 4), and an odd square is of the form 1 (mod 4).

We had observed above that a, b have the same parity and c, d have the same parity. Suppose that a, b are both odd and c, d are both even; then the relation $a^2 + b^2 = k$ shows that $k \equiv 2 \pmod 4$, while the relation $c^2 + d^2 = k$ shows that $k \equiv 0 \pmod 4$. We see a contradiction here. The same contradiction arises if we suppose that a, b are both even and c, d are both odd. We are forced to conclude that a, b, c, d all have the same parity. As we have assumed that at least one of them is an

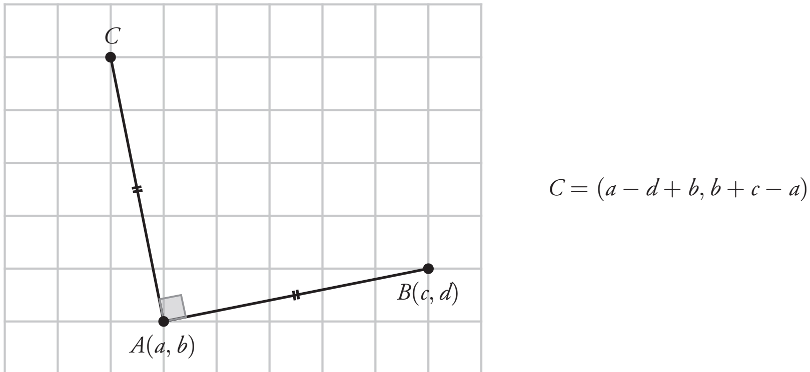


Figure 3

odd number, it must be that a, b, c, d are all odd. Hence $a - c$ and $b - d$ are both even.

The relation $k = a^2 + b^2$ now tells us that k must be of the form 2 (mod 4). On the other hand, the relation $k = (a - c)^2 + (b - d)^2$ tells us that k must be of the form 0 (mod 4).

We have thus arrived at a contradiction, and this shows that it is not possible to find a lattice-point equilateral triangle. □

Fourth proof: An argument based on descent. Our fourth (and last) proof is subtler than the earlier ones. However, it uses important mathematical ideas and is worth studying deeply. Its basis lies in a fundamental symmetry of the lattice points of the coordinate plane: if about any lattice point as centre we perform a 90° rotation (either clockwise or anticlockwise), then lattice points get mapped to lattice points, and non-lattice points get mapped to non-lattice points. This may also be checked using simple computations: a 90° anticlockwise rotation about point $A(a, b)$ will take point $B(c, d)$ to point C where $C = (a - d + b, b + c - a)$. If a, b, c, d are integers, then obviously $a - d + b$ and $b + c - a$ are integers. So if A and B are lattice points, then C too is a lattice point. (See Figure 3.)

Now let us suppose that there exists a lattice-point equilateral triangle ABC. Figure 4 depicts the situation. Consider the effect of a 90° rotation (anticlockwise) about point A. Let the rotation take point B to point D. As per what we said above, D must be a lattice point.

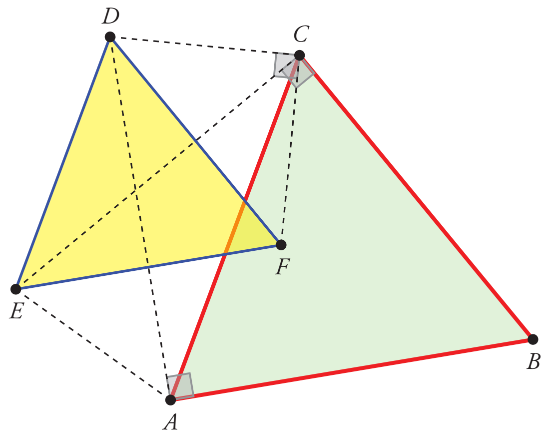


Figure 4

In the same way, let a 90° rotation (clockwise) be performed about point C as centre. Let this take point B to point E. Then E too is a lattice point. Finally, consider the effect of a 90° rotation about point C (anticlockwise). Let it take point D to point F. Then F too is a lattice point. Hence $\triangle DEF$ is a lattice-point triangle. We shall show that DEF is an equilateral triangle.

We start by noting that $\triangle CAD$ and $\triangle ACE$ are congruent isosceles triangles, each with apex angle 30°. Hence $AE = CD$, and $AEDC$ is an isosceles trapezium, with $ED \parallel AC$ and $\measuredangle ACD = \measuredangle CAE$. Since $\measuredangle ACD = 75^\circ$, it follows that $\measuredangle CDE = 105^\circ$.

Again, $CD = CF$ and $\measuredangle DCF = 90^\circ$, so $\measuredangle CDF = 45^\circ$. Hence $\measuredangle EDF = 60^\circ$.

Next, observe that $\measuredangle DCE = 75^\circ - 30^\circ = 45^\circ$. Since $\measuredangle CDF = 45^\circ$ as well, it follows that

$CE \perp DF$. Since $CD = CF$, this means that CE bisects DF at right angles. Hence E is equidistant from D and F, i.e., $ED = EF$. Hence $\measuredangle EFD = \measuredangle EDF$, i.e., $\measuredangle EFD = 60^\circ$. It follows that $\triangle DEF$ is equilateral. So $\triangle DEF$ is a lattice-point equilateral triangle.

Now let us compare the sizes of these two equilateral triangles. We have, from $\triangle CDF$:

$$\frac{DF}{CD} = \frac{1}{\sin 45^\circ}.$$

Next, from $\triangle ACD$:

$$\frac{CD}{AC} = \frac{\sin 30^\circ}{\sin 75^\circ}.$$

Hence:

$$\frac{DF}{AC} = \frac{1}{\sin 45^\circ} \times \frac{\sin 30^\circ}{\sin 75^\circ} = \frac{\sin 45^\circ}{\sin 75^\circ},$$

since $\sin^2 45^\circ = \sin 30^\circ$. Since $\sin 45^\circ$ is smaller than $\sin 75^\circ$, it follows that $DF < AC$. (In fact, $\sin 45^\circ / \sin 75^\circ = \sqrt{3} - 1 \approx 0.732 < 0.75 < 1$.)

Hence the equilateral triangle DEF is *strictly smaller* than the equilateral triangle ABC: its sides are shorter than $3/4$ of the sides of the original triangle. Thus, by following the geometrical procedure described above, we have generated a new lattice-point equilateral triangle whose sides are shorter than $3/4$ of the sides of the original triangle.

By applying the same procedure to $\triangle DEF$, we generate another lattice-point equilateral triangle GHI (say), whose sides are shorter than $3/4$ of the sides of $\triangle DEF$. And we can continue in this way, generating a sequence of lattice-point equilateral triangles whose sides are decreasing in a geometrical ratio which is strictly less than 1. But this is clearly impossible, because after a while we will obtain lattice-point triangles whose sides are less than 1 in length! However, the distance between two lattice points obviously cannot be less than 1. Hence such a sequence of triangles cannot exist.

It follows that a lattice-point equilateral triangle does not exist. □

Remarks. As noted earlier, four proofs for the same result may seem like an overkill; but not if they illustrate important mathematical ideas, and that is certainly true of the proofs we have described. Each one is distinctive in its own way, though the first two have the common feature that they both depend on the irrationality of $\sqrt{3}$. The first three arguments are number theoretic, while the fourth proof is of a completely different nature.

How close can we come to finding a lattice-point equilateral triangle?

Having shown the impossibility of some phenomenon (in this case, the existence of a lattice-point equilateral triangle), we naturally want to know how close we can get to it. First we need a measure to assess how close is 'close'. We could choose a measure based on side-lengths, or one based on angles. Here is a possible measure of the discrepancy between a given triangle and an equilateral triangle, based on side-lengths: if the triangle has side-lengths a, b, c, then we compute the quantity q given by

$$q = \frac{ab + bc + ca}{a^2 + b^2 + c^2}.$$

It is a nice exercise (please try it) to show that if a, b, c are any three real numbers, then

$$ab + bc + ca \leq a^2 + b^2 + c^2,$$

with equality precisely when $a = b = c$. It follows that the computed quantity satisfies the inequality $0 < q \leq 1$, and equality holds precisely when the triangle is equilateral. Hence the gap between q and 1 is a measure of how far the triangle is from being equilateral.

We can now embark on a search for lattice-point triangles for which the q-value is very close to 1. How do we conduct such a search? Purely empirically, using a computer to check through millions of possibilities? Or is there a nicer way than that? Let us defer this question to the next part of this article. In the meantime, why don't we pass on the question to you? Please see how close you can come to finding a lattice-point equilateral triangle.

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