The Constants of Mathematics

Part 3

Features

More on the Remarkable Number e

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In this article, which is the third of our series on mathematical constants, we continue our exploration of Euler's constant e. (Yes, we have had to devote more than one 'episode' to e, as there is so much to say about this number.)

More infinite series for e

In the previous part of this article, we pointed out that a simple consequence of the definition $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$ is the following infinite series for e:

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}. \qquad (1)$$

We mentioned at the time that this infinite series converges quite rapidly. Let $B(n)$ denote the sum $1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$; then we have the following data:

n	$e - B(n)$
10	2.73×10^{-8}
20	2.05×10^{-20}
30	1.26×10^{-34}

Now here's the surprise: by making an apparently minor tweak to the series (1), we can increase the rate of convergence quite dramatically! Here is how we do it.

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A tweaked series. The idea is very simple indeed: all we do is to combine adjacent pairs of terms. Thus we have:

$$\begin{aligned} e &= \left(\frac{1}{0!}+\frac{1}{1!}\right)+\left(\frac{1}{2!}+\frac{1}{3!}\right)+\left(\frac{1}{4!}+\frac{1}{5!}\right)+\cdots \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{(2n)!}+\frac{1}{(2n+1)!}\right) \\ &= \sum_{n=0}^{\infty}\left(\frac{2n+1}{(2n+1)!}+\frac{1}{(2n+1)!}\right) \\ &= \sum_{n=0}^{\infty}\frac{2n+2}{(2n+1)!}. \end{aligned}$$

So we have the following result:

$$e = \frac{2}{1!}+\frac{4}{3!}+\frac{6}{5!}+\frac{8}{7!}+\cdots = \sum_{n=0}^{\infty}\frac{2n+2}{(2n+1)!}. \tag{2}$$

Let $C(n)$ denote the sum $\frac{2}{1!}+\frac{4}{3!}+\frac{6}{5!}+\cdots+\frac{2n+2}{(2n+1)!}$; then we have the following data:

n	$e - C(n)$
10	9.30×10^{-22}
20	7.29×10^{-52}
30	3.23×10^{-86}

A hugely faster rate of convergence! Similarly, we have the following result:

$$e = 1+\frac{3}{2!}+\frac{5}{4!}+\frac{7}{6!}+\frac{9}{8!}+\cdots = \sum_{n=0}^{\infty}\frac{2n+1}{(2n)!}. \tag{3}$$

This series too converges more rapidly than (1), but not as rapidly as (2). We leave the proof of (3) for you to find.

Yet more such tweaks are possible, by bracketing more terms together. Thus we have:

$$e = \sum_{n=0}^{\infty}\frac{(3n)^2+1}{(3n)!} = \frac{1}{0!}+\frac{3^2+1}{3!}+\frac{6^2+1}{6!}+\frac{9^2+1}{9!}+\frac{12^2+1}{12!}+\cdots. \tag{4}$$

This series converges even more rapidly than (2). Numerous such results are possible.

Simple continued fraction for ***e***

In the previous part of this article, we had pointed out that the following fraction is a very good rational approximation for e:

$$\frac{23225}{8544},$$

which differs from e by roughly 6.7×10^{-9}. We had asked how such rational approximations can be found and noted that they come from the 'simple continued fraction' for e. We now elaborate on this comment.

First we explain what is meant by a *simple continued fraction* (SCF). This is best done by means of a few examples. Below, we express the fractions $\frac{7}{3}$, $\frac{7}{5}$ and $\frac{11}{7}$ as SCFs:

$$\frac{7}{3} = 2 + \frac{1}{3},$$

$$\frac{7}{5} = 1 + \cfrac{1}{2 + \cfrac{1}{2}},$$

$$\frac{11}{7} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3}}}.$$

Observe that they are cumbersome to write! For this reason, short forms are used which consume less space and also are easier to typeset. Here is one short form which is used:

$$\frac{7}{3} = [2; 3], \qquad \frac{7}{5} = [1; 2, 2], \qquad \frac{11}{7} = [1; 1, 1, 3].$$

The examples shown above are *finite* SCFs. It is easy to show that every rational number can be expressed as a finite SCF. (There are precisely two finite SCFs corresponding to each irrational number. However, they differ in a rather inconsequential way.)

It follows immediately that *an infinite SCF must correspond to an irrational number*. One of the simplest and most elegant examples of this is the SCF for the golden ratio ϕ:

$$\phi = \frac{1}{2}\left(\sqrt{5} + 1\right) = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}} = [1; 1, 1, 1, 1, 1, \ldots].$$

Here the SCF is made up solely of 1's.

It was Euler who found the SCF corresponding to e. It is a result of great beauty:

$$2 = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}}}}}}}}, \tag{5}$$

i.e.,

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, \ldots].$$

Observe the sequence of denominators:

$$1,\ 2,\ 1,\ 1,\ 4,\ 1,\ 1,\ 6,\ 1,\ 1,\ 8,\ 1,\ 1,\ 10,\ 1,\ \ldots.$$

However, it will not be possible for us to give the proof of this result here; it is *way* beyond the scope of this article. Interested readers can refer to [1] or [6] for the proof.

Using the SCF to find a good rational approximation for *e*. The standard theory behind infinite SCFs tells us that if we truncate the SCF at any high denominator and compute the resulting finite SCF, the answer will be very close to the value of the infinite SCF. Here, let us compute the values of the SCFs obtained by truncating the infinite SCF at 6, 8 and 10, respectively; we get:

$$[2; 1, 2, 1, 1, 4, 1, 1, 6] = \frac{1264}{465},$$

$$[2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8] = \frac{23225}{8544},$$

$$[2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10] = \frac{517656}{190435}.$$

It follows that the fractions

$$\frac{1264}{465},\quad \frac{23225}{8544}\quad \text{and}\quad \frac{517656}{190435}$$

are progressively better rational approximations for e. (The fraction in the middle is the one we had exhibited earlier.) This answers the question we had raised in the earlier part of the article.

For more general information about continued fractions, the reader could refer to [7].

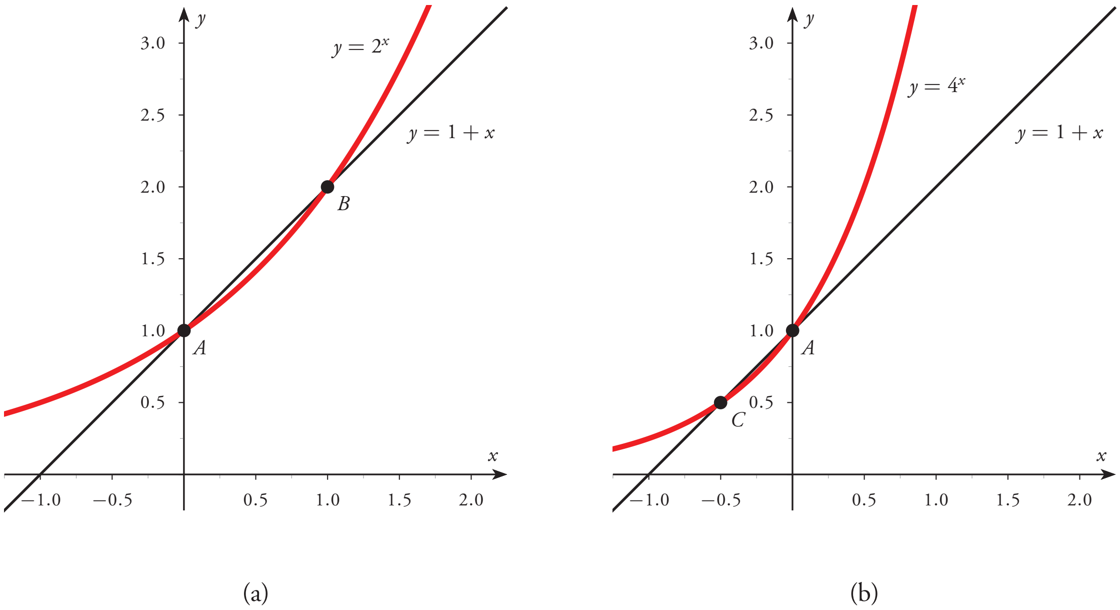
Difference between a SCF and a GCF. The significance attached to the word 'simple' in 'SCF' is that the numerators in the continued fraction are all 1's. If we relax this condition, we get constructs which are called *general continued fractions* or GCFs. For example:

$$\frac{268}{113} = 2 + \cfrac{3}{7 + \cfrac{5}{4 + \cfrac{2}{3}}}.$$

We have the following astonishing result—an infinite GCF for e:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \cfrac{5}{6 + \cfrac{6}{7 + \cfrac{7}{8 + \cfrac{8}{9 + \cfrac{9}{\cdots}}}}}}}}}}. \qquad (6)$$

How beautiful this result looks!



Graphs of $y = 1 + x$, $y = 2^x$ and $y = 4^x$

The only number satisfying a certain inequality

Consider the inequality $2^x \geq 1 + x$. Is this true for all real values of x? If we draw the graphs corresponding to $y = 2^x$ and $y = 1 + x$, we find that they intersect at $x = 0$ and $x = 1$. We observe from the graph that the inequality $2^x > 1 + x$ holds (strictly) for $x < 0$ and for $x > 1$. But for $0 < x < 1$, the inequality is reversed; we have $2^x < 1 + x$. So it is *not true* that $2^x \geq 1 + x$ for all real values of x; the inequality is falsified over the interval $0 < x < 1$. See Figure 1 (a).

Next, consider the inequality $4^x \geq 1 + x$. Is this true for all real values of x? Probing as we did earlier, we find that the graphs of $y = 4^x$ and $y = 1 + x$ intersect at $x = -0.5$ and $x = 0$; so the inequality is falsified over the interval $-0.5 < x < 0$. Outside of this interval, the inequality $4^x \geq 1 + x$ is valid. See Figure 1 (b).

How about the inequality $3^x \geq 1 + x$? Is this true for all real values of x? Finding the points of intersection of these two graphs involves more computation, but after some effort we find that there are intersection points at $x = 0$ and $x = -0.174$ (approximately); so the inequality is falsified over the interval $-0.174 < x < 0$. Outside of this interval, the inequality $3^x \geq 1 + x$ is valid.

How about the inequality $2.5^x \geq 1 + x$. Is this true for all real values of x? Once again, finding the points of intersection of the two graphs involves a fair bit of computation, but we find that there are intersection points at $x = 0$ and $x = 0.188$ (approximately); so the inequality is falsified over the interval $0 < x < 0.188$. Outside of this interval, the inequality $2.5^x \geq 1 + x$ is valid.

It is tempting to draw a conjecture from this pattern: namely, that for each real number $a > 1$, the inequality $a^x \geq 1 + x$ is never valid for the entire set of all real numbers x; that there is always some interval where the inequality is falsified. *But this conjecture is wrong!* It turns out that there is one (and precisely one) real number $a > 1$ for which it is true that $a^x \geq 1 + x$ for all real values of x, and that number is $a = e$.

In other words, the following claims are true:

(i) If $a > 1$ and $a \neq e$, then there exist real values of x for which $a^x < 1 + x$.

(ii) The inequality $e^x \geq 1 + x$ holds for all real values of x.

Proof of (i). We use the fact that *the only positive number a for which the graph of* $y = a^x$ *has slope* 1 *at* $x = 0$ *is* $a = e$. (This was proved in Part 2 of this series of articles. As noted there, this property can be used to *define* e.) Moreover, the following is true: if $a > e$, then the slope of $y = a^x$ at $x = 0$ is greater than 1, and if $0 < a < e$, then the slope of $y = a^x$ at $x = 0$ is less than 1. Let us see how these two facts imply claim (ii).

- Suppose that $a > e$. Then the slope of $y = a^x$ at $x = 0$ is greater than 1. This means that at the point $(0, 1)$, *the curve crosses the line* $y = 1 + x$ *from below to above*; i.e., the curve lies *below* the line in the region immediately to the left of $x = 0$, and it lies *above* the line in the region immediately to the right of $x = 0$. *This implies that* $a^x < 1 + x$ *in some region immediately to the left of* $x = 0$. In other words, there exists some negative number c, whose value naturally will depend on a, such that for $c < x < 0$, we have $a^x < 1 + x$. (Figure 1 (b) may make this clearer.)
- Suppose that $0 < a < e$. Then the slope of $y = a^x$ at $x = 0$ is less than 1. This means that at the point $(0, 1)$, *the curve crosses the line* $y = 1 + x$ *from above to below*; i.e., the curve lies *above* the line in the region immediately to the left of $x = 0$, and it lies *below* the line in the region immediately to the right of $x = 0$. *This implies that* $a^x < 1 + x$ *in some region immediately to the right of* $x = 0$. In other words, there exists some positive number d, whose value naturally will depend on a, such that for $0 < x < d$, we have $a^x < 1 + x$. (Figure 1 (a) may make this clearer.)

So statement (i) has been proved.

Proof of (ii). We first note that the fact that

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

implies that the derivative of e^x is e^x. (This is a well-known fact, and it is studied in the +2 mathematics course. But for the sake of completeness, we include the proof here.) To see why, note that (by definition) the slope of the curve $y = e^x$ at the point $P\ (a, e^a)$ is

$$\lim_{h\to 0} \frac{e^{a+h} - e^a}{h} = e^a \cdot \lim_{h\to 0} \frac{e^h - 1}{h} = e^a \cdot 1 = e^a.$$

We now show how this result can be used to prove that $e^x \geq 1 + x$ for all real values of x.

Define a function g on the set of real numbers $\mathbb{R}$ as follows:

$$g(x) = e^x - 1 - x.$$

The derivative of g is $g'(x) = e^x - 1$. Since $e > 1$, the following statements are true:

- If $x < 0$, then $e^x < 1$, hence $g'(x) < 0$.
- If $x > 0$, then $e^x > 1$, hence $g'(x) > 0$.

It follows that $g(x)$ *is strictly decreasing when* $x < 0$, *and* $g(x)$ *is strictly increasing when* $x > 0$. Hence $g(x)$ achieves its global minimum at $x = 0$, i.e., $g(x) \geq g(0)$ for all x.

Since $g(0) = 0$, it follows that $g(x) \geq 0$ for all x, i.e., $e^x \geq 1 + x$ for all x.

Appendix

In the previous part of the article, we considered the curve $y = \frac{1}{x}$ and defined a function $f(t)$ for $t > 0$ thus: $f(t)$ = the area enclosed by the curve, the x-axis and the lines $x = 1$ and $x = t$. Then f is a continuous function, and $f(1) = 0$. We found, using simple computation, that $f(2) < 1$ and $f(3) > 1$. By continuity, there exists a value of t between 2 and 3 such that $f(t) = 1$. We claimed that this critical value is e. Let us give here the steps needed to prove this claim. (We shall leave the individual steps as problems for the reader.)

Step 1: Show that if $a > 0$ and $b > 0$, then $f(ab) = f(a) + f(b)$.

Step 2: Deduce that for any $a > 0$ and any positive integer n, $f(a^n) = n \cdot f(a)$.

Step 3: Show that for any positive integer n,

$$\frac{1}{n+1} < f\left(1 + \frac{1}{n}\right) < \frac{1}{n}.$$

Step 4: Deduce that for any positive integer n,

$$\frac{n}{n+1} < n \cdot f\left(1 + \frac{1}{n}\right) < 1,$$

and hence that

$$\frac{n}{n+1} < f\left(\left(1 + \frac{1}{n}\right)^n\right) < 1.$$

Step 5: In the above relation, let $n \to \infty$; the quantity on the extreme left then tends to 1, and the quantity on the extreme right is equal to 1. The 'sandwich principle' now applies, and we deduce that

$$\lim_{n\to\infty} f\left(\left(1 + \frac{1}{n}\right)^n\right) = 1.$$

Step 6: Deduce from the above that the value of t such that $f(t) = 1$ is equal to

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

But this number is e, by definition. This proves the claim that had been made.

Remark. We have yet to exhaust the list of remarkable features that e possesses!

References

1. Cohn, Henry; A short proof of the simple continued fraction expansion of e, https://arxiv.org/abs/math/0601660
2. The number *e*, http://www-history.mcs.st-and.ac.uk/HistTopics/e.html
3. Maor, Eli; *e: The Story of a Number*, ISBN 0-691-05854-7
4. Sondow, Jonathan and Weisstein, Eric W. "e." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/e.html
5. Wikipedia, *e* (mathematical constant), https://en.wikipedia.org/wiki/E_(mathematical_constant)
6. Wikipedia, Euler's continued fraction formula, https://en.wikipedia.org/wiki/Euler%27s_continued_fraction_formula
7. Wikipedia, Continued fraction, https://en.wikipedia.org/wiki/Continued_fraction

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