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Finding the Number of Ordered Tuples Having a Given LCM

RAHIL MIRAJ

Counting is one of the first skills that a student of mathematics learns. Here we feature an article that observes patterns while counting and generalises this pattern. While the actual combinatorics and use of the binomial theorem may be appreciated only by students of classes 11 and 12, students of High School will certainly be able to follow the reasoning in the two examples given. It is important for students to have such gentle introductions to mathematical notation and theorems used at a more senior level.

In this article, we derive a formula for the number of ordered n-tuples of positive integers whose LCM is a given integer. More specifically, we prove the following theorem.

Theorem. *The number of ordered n-tuples whose LCM is $p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$, where $p_1, p_2, \cdots, p_m$ are prime numbers and $a_1, a_2, \cdots, a_m$ are non-negative integers, is equal to*

$$\prod_{i=1}^{m} \left[(a_i + 1)^n - a_i^n\right]. \tag{1}$$

Before proceeding with the proof of this claim, we illustrate the use of the formula with some examples.

Example 1. To find the number of ordered pairs (a, b) of positive integers whose LCM is $p^2 q^4 r^2$, where p, q, r are prime numbers.

Solution. No prime number other than p, q, r can divide either a or b. Let us write

$$a = p^u q^v r^w, \quad b = p^x q^y r^z,$$

where u, v, w, x, y, z are non-negative integers. Since the LCM of a, b is $p^2 q^4 r^2$, it must be true that

$$u, x \in \{0, 1, 2\}, \quad v, y \in \{0, 1, 2, 3, 4\}, \quad w, z \in \{0, 1, 2\},$$

and

$$\max(u, x) = 2, \quad \max(v, y) = 4, \quad \max(w, z) = 2.$$

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The ordered pair (u, x) can be any of the following (5 possibilities):

$$(0,2),\ (2,0),\ (1,2),\ (2,1),\ (2,2).$$

The ordered pair (v, y) can be any of the following (9 possibilities):

$$(0,4),\ (4,0),\ (1,4),\ (4,1),\ (2,4),\ (4,2),\ (3,4),\ (4,3),\ (4,4).$$

The ordered pair (w, z) can be any of the following (5 possibilities):

$$(0,2),\ (2,0),\ (1,2),\ (2,1),\ (2,2).$$

Therefore, using multiplication principle, the number of ordered pairs (a, b) is

$$5 \times 9 \times 5 = 225.$$

Now we solve the same problem by using the theorem. Here,

$$m = 3,\ n = 2,\ a_1 = 2,\ a_2 = 4,\ a_3 = 2.$$

Hence the required number is

$$\begin{aligned}&\prod_{i=1}^{3} \left[(a_i+1)^2 - a_i^2\right] \\ &= \left[3^2 - 2^2\right] \cdot \left[5^2 - 4^2\right] \cdot \left[3^2 - 2^2\right] \\ &= 5 \times 9 \times 5 = 225.\end{aligned}$$

Example 2. To find the number of ordered triples (a, b, c) of positive integers whose LCM is $p^2q^4r^3s^2$, where p, q, r, s are prime numbers.

Solution. No prime number other than p, q, r, s can divide a, b, c. Let us write

$$a = p^u q^v r^w s^x, \quad b = p^{u'} q^{v'} r^{w'} s^{x'}, \quad c = p^{u''} q^{v''} r^{w''} s^{x''},$$

where $u, v, w, x, u', v', w', x', u'', v'', w'', x''$ are non-negative integers. Since the LCM of a, b, c is $p^2q^4r^3s^2$, it must be true that

$$u, u', u'' \in \{0,1,2\}, \quad v, v', v'' \in \{0,1,2,3,4\}, \quad w, w', w'' \in \{0,1,2,3\}, \quad x, x', x'' \in \{0,1,2\},$$

and

$$\max(u,u',u'') = 2, \quad \max(v,v',v'') = 4, \quad \max(w,w',w'') = 3, \quad \max(x,x',x'') = 2.$$

The ordered triple (u, u', u'') can be any of the following (19 possibilities):

$$\begin{array}{lllllll}(2,0,0), & (2,1,0), & (2,0,1), & (2,1,1), & (2,2,0), & (2,0,2), & (2,1,2), \\ (2,2,1), & (2,2,2), & (0,2,0), & (0,2,1), & (1,2,0), & (1,2,1), & (0,2,2), \\ (1,2,2), & (0,0,2), & (1,0,2), & (0,1,2), & (1,1,2). & & \end{array}$$

The ordered triple (v, v', v'') can be any of the following (61 possibilities):

$$\begin{array}{lllllll}(0,0,4), & (0,4,0), & (4,0,0), & (1,1,4), & (1,4,1), & (4,1,1), & (2,2,4), \\ (2,4,2), & (4,2,2), & (3,3,4), & (3,4,3), & (4,3,3), & (4,4,4), & (0,1,4), \\ (1,0,4), & (0,2,4), & (2,0,4), & (0,3,4), & (3,0,4), & (1,2,4), & (2,1,4), \\ (1,3,4), & (3,1,4), & (2,3,4), & (3,2,4), & (0,4,1), & (1,4,0), & (0,4,2), \\ (2,4,0), & (0,4,3), & (3,4,0), & (1,4,2), & (2,4,1), & (1,4,3), & (3,4,1), \\ (2,4,3), & (3,4,2), & (4,0,1), & (4,1,0), & (4,0,2), & (4,2,0), & (4,0,3), \\ (4,3,0), & (4,1,2), & (4,2,1), & (4,1,3), & (4,3,1), & (4,2,3), & (4,3,2), \\ (4,4,0), & (4,4,1), & (4,4,2), & (4,4,3), & (4,0,4), & (4,1,4), & (4,2,4), \\ (4,3,4), & (0,4,4), & (1,4,4), & (2,4,4), & (3,4,4). & & \end{array}$$

The ordered triple (w, w', w'') can be any of the following (37 possibilities):

$$\begin{array}{lllllll}
(0,0,3), & (0,3,0), & (3,0,0), & (1,1,3), & (1,3,1), & (3,1,1), & (2,2,3), \\
(2,3,2), & (3,2,2), & (3,3,3), & (0,1,3), & (1,0,3), & (0,2,3), & (2,0,3), \\
(1,2,3), & (2,1,3), & (0,3,1), & (1,3,0), & (0,3,2), & (2,3,0), & (1,3,2), \\
(2,3,1), & (3,0,1), & (3,1,0), & (3,0,2), & (3,2,0), & (3,1,2), & (3,2,1), \\
(3,3,0), & (3,3,1), & (3,3,2), & (3,0,3), & (3,1,3), & (3,2,3), & (0,3,3), \\
(1,3,3), & (2,3,3). & & & & &
\end{array}$$

The ordered triple (x, x', x'') can be any of the following (19 possibilities):

$$\begin{array}{lllllll}
(2,0,0), & (2,1,0), & (2,0,1), & (2,1,1), & (2,2,0), & (2,0,2), & (2,1,2), \\
(2,2,1), & (2,2,2), & (0,2,0), & (0,2,1), & (1,2,0), & (1,2,1), & (0,2,2), \\
(1,2,2), & (0,0,2), & (1,0,2), & (0,1,2), & (1,1,2). & &
\end{array}$$

Therefore, using multiplication principle, the number of ordered triples (a, b, c) is

$$19 \times 61 \times 37 \times 19 = 814777.$$

Now we solve the same problem by using the theorem. Here,

$$m = 4, \ \ n = 3, \ \ a_1 = 2, \ \ a_2 = 4, \ \ a_3 = 3, \ \ a_4 = 2.$$

Hence the required number is

$$\begin{aligned}
&\prod_{i=1}^{4} \left[(a_i + 1)^3 - a_i^3\right] \\
&= \left[3^3 - 2^3\right] \cdot \left[5^3 - 4^3\right] \cdot \left[4^3 - 3^3\right] \cdot \left[3^3 - 2^3\right] \\
&= 19 \times 61 \times 37 \times 19 = 814777.
\end{aligned}$$

Proof of the theorem

For convenience, we repeat the statement of the theorem:

Theorem. *The number of ordered n-tuples whose LCM is $p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$, where $p_1, p_2, \cdots, p_m$ are prime numbers and $a_1, a_2, \cdots, a_m$ are non-negative integers, is equal to*

$$\prod_{i=1}^{m} \left[(a_i + 1)^n - a_i^n\right]. \tag{2}$$

Let $(b_1, b_2, \ldots, b_n)$ be an ordered n-tuple of positive integers whose LCM is

$$p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}.$$

We must count the number of such n-tuples. Let

$$b_j = p_1^{c_{1j}} \cdot p_2^{c_{2j}} \cdots p_m^{c_{mj}} \qquad (1 \leq j \leq n). \tag{3}$$

We must have:

$$c_{11}, c_{12}, \ldots c_{1n} \in \{0, 1, 2, \ldots, a_1\}, \quad \max\left(c_{11}, c_{12}, \ldots c_{1n}\right) = a_1. \tag{4}$$

How many possibilities are there for the n-tuple $(c_{11}, c_{12}, \ldots c_{1n})$, subject to (4)? At least one c_{1j} must be at its maximum possible value (a_1). Let us fix these c_{1j}'s at the start; let k of the c_{1j}'s be at their maximum possible value ($1 \leq k \leq n$). These c_{1j}'s can be chosen in $\binom{n}{k}$ possible ways. Each of the remaining c_{1j}'s can take a_1 possible values (i.e., all the values from 0 to $a_1 - 1$). Hence the number of choices for the remaining c_{1j}'s is a_1^{n-k}. Hence the number of possibilities for the n-tuple $(c_{11}, c_{12}, \dots c_{1n})$, subject to (4), is

$$\sum_{k=1}^{n} \binom{n}{k} \cdot a_1^{n-k}.$$

By the binomial theorem, this quantity is equal to

$$(a_1 + 1)^n - a_1^n.$$

The same reasoning holds for the other tuples. Hence, by the multiplication principle, the number of possible ordered n-tuples $(b_1, b_2, \dots, b_n)$ is equal to

$$\left[(a_1 + 1)^n - a_1^n\right] \cdot \left[(a_2 + 1)^n - a_2^n\right] \cdot \dots \cdot \left[(a_m + 1)^n - a_m^n\right], \tag{5}$$

i.e., $\prod_{i=1}^{m} \left[(a_i + 1)^n - a_i^n\right]$.

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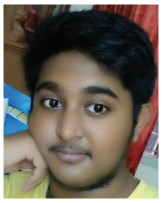

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