Cutting a Square into Equal Parts

VINAY NAIR

Pre-requisites

The article talks about a simple activity which can be performed with students of primary, middle and high school. The shape that is used to discuss here is a square and hence it is expected that students know the basic properties of a square. The article also talks about using lines. Even if students don't have a Euclidean notion of definition of a line, that idea can be instilled as the teacher executes this activity. Similarly, students might have a notion of the words *equal, area, congruence, similar*, etc., and this activity is very useful in clarifying the subtle difference between terminologies as well as the rigour of mathematical language.

Another important aspect is that the activity allows the teacher to introduce concepts like countable, uncountable and infinite (not in the notion of different types of infinity, but in the sense of a layman's usage of the words). This is a very fruitful exercise because many a time all these words are mixed up in the usage by children.

Towards the end of the article, inside a table, the reader will find the question ***Why*** asked quite a few times. This is *why* the activity is useful for high school students as well because they will be in a better position to reason as to why they believe the answer is so for a particular number of lines. If they have to give reasons, they might have to take a dip into the axioms and definitions of certain concepts, which they don't do otherwise while solving school problems. At primary and middle school level also the reasoning can (and should) be asked, but at that age, they might not be introduced to the idea of axioms and building theories in Mathematics by changing axioms.

Last, but not the least, the activity can be explored using other shapes as well and a comparative study made between different shapes.

Keywords: Square, symmetry, equality, congruency, area, generalisation

"In how many ways can you divide a square into equal parts using a line?" is a question that I have often asked middle school students. Interestingly, I get new answers every time I ask this question and the questions that follow. The most common answer is 'Four,' the four ways being:

Category - I

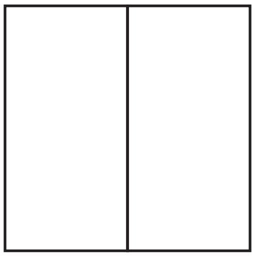

Figure 1.1

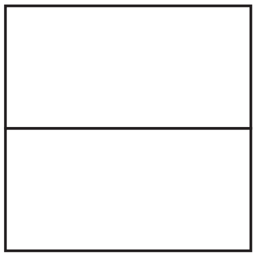

Figure 1.2

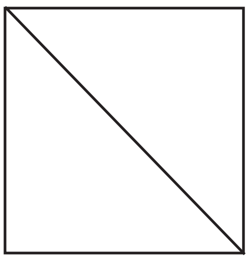

Figure 1.3

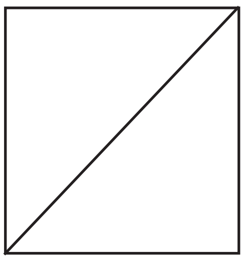

Figure 1.4

I imagine the reason why most people think of these four ways is because they would have done something similar in their own lives through paper folding (cutting out a square from a rectangular sheet of paper), opening and closing books, cutting cakes or chocolates, folding bedsheets/mats, etc. One can think of so many situations in which such folding must have been done.

Some students come up with non-routine solutions, like:

Category - II

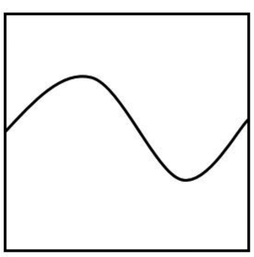

Figure 2.1

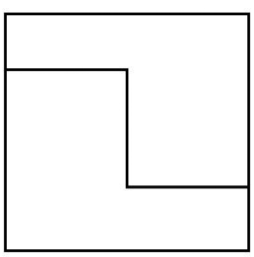

Figure 2.2

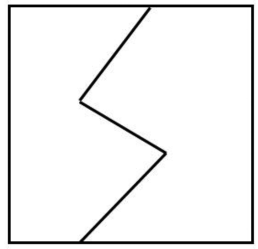

Figure 2.3

Surprisingly, such solutions usually don't come from the older kids who have been conditioned by thinking patterns; and we rarely see adults making such designs. When students come up with such designs, as a teacher one might be tempted to say that they are *wrong* as the question clearly tells us that we must use *a* ***line***, and that line should be always **straight**. But this is a good way to kill the creativity of a student. Instead, when we ask them to explain what they mean by *lines*, the students who have drawn this understand these dividers also as a line probably because they might have seen queues outside a ticketing office that go in these shapes which we call colloquially as *lines!*

Most of the students in the class don't accept the divisions in category II. This is an opportunity for the teacher to discuss in the class - *What is a line?* I've heard a nice definition from many students for a *straight line* - A straight line is a line that doesn't bend. If we can avoid the mathematical rigour of defining a line, we can probably accept this definition. Usually, at this point, the majority of the class agrees that there are four ways of dividing a square into equal parts using a line if we consider divisions under Category - I, and infinite[1] ways if we consider Category - II. At this juncture, I tell them that even in Category - I, there are more than four ways of dividing it and challenge them to find more ways. Subsequently, I have seen a few students drawing the image below:

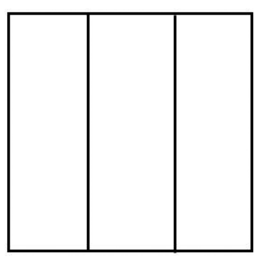

Figure 2.4

When I have asked them to read the question again, I have found out that they have taken the phrase *'…a line'* in the question to be *'…one line.'* When this is clarified, they take back their answer. However, there is always some kid who thinks of the idea sketched below:

Category - III

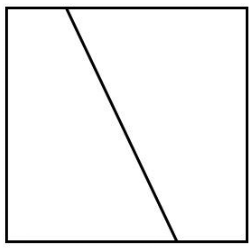

Figure 3.1

There is a general confusion amongst students when you ask them if the parts shown are equal parts. Many say, *'Yes'*, some say, *'No'*. When the latter are asked the reason for their answer, most of them say that they don't seem to be equal. One of the interesting answers that I got from a student recently was - *'How can they be equal if they are not symmetric?,* (By *symmetry*, the student was referring to reflective symmetry.) So I asked the class if they thought it is true that if two things have to be equal, they should have reflective symmetry as well. After some discussion, a few of them came up with counter examples where two things were equal even when they did not have reflective symmetry. Thus, this new category of dividing a square into equal parts is added to the answer. But there are still a few who want to be convinced that the two parts are equal. So they are asked to construct a square and draw a line from a point on one edge in such a way that when the line meets the opposite edge, the length from corner to starting point is the same as that from the opposite corner of the square to the end point. And then, measure all the sides and angles of the two shapes formed; or cut out the shapes and see if they fit exactly one on top of each other. Again, the question still remains - how many such possibilities exist? Most say, *'Two'* and then say *'Four'*.

Category - III

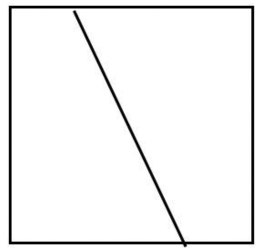

Figure 3.1

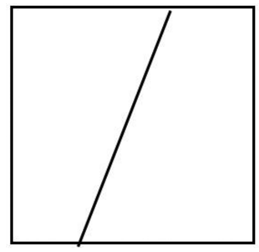

Figure 3.2

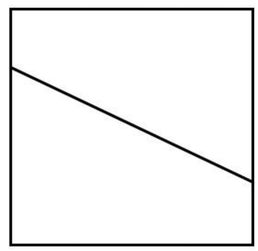

Figure 3.3

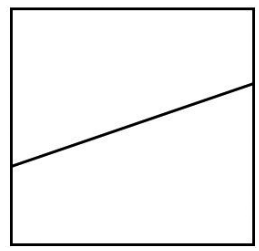

Figure 3.4

But in no time, someone else comes up saying that more ways are possible because you can alter the distance where the divider meets the sides of the square and still cut it into equal parts. Thus, the class agrees that there are **many, many** ways to cut a square into equal parts using a line.

Cutting a square into equal parts using two lines

The next question that follows is –*"In how many ways can you divide a square into equal parts using two lines?"* The following two answers come up:

Category - IV

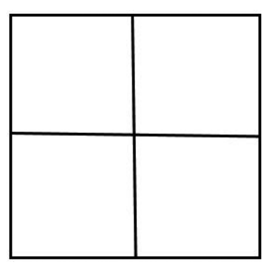

Figure 4.1

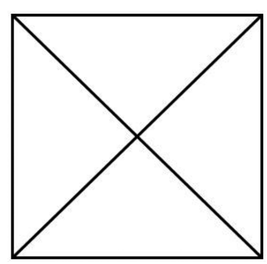

Figure 4.2

Category - V

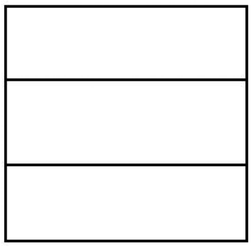

Figure 5.1

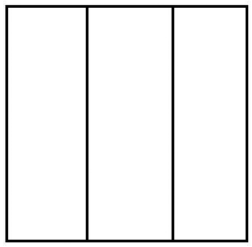

Figure 5.2

In most cases, students come up with either Category IV or V and not both. This is an interesting observation. Naturally, the next question is - *How many ways are there to cut a square into four equal parts and three equal parts using two lines?* Students start working on it immediately.

If you would have noticed, until the last question, we have never mentioned the number of equal parts into which they are supposed to divide the square. Everywhere it says, 'divide the square into equal parts.' Why? When we don't give them the number of equal parts, we are not conditioning them or putting any constraint on them in thinking. This frees their imagination which is very important to bring about creativity through mathematics.

Let us come back to the question. Most students who would have explored say that there are only two ways in Category IV and two ways in Category V. However, there have always been two or three students out of a class of 30-35, who draw the image below and say that they have found a new way.

Category - VI (a)

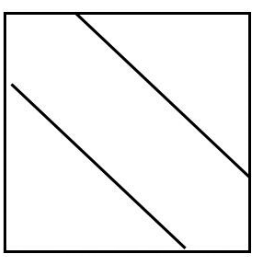

Figure 6.1

Category - VI (b)

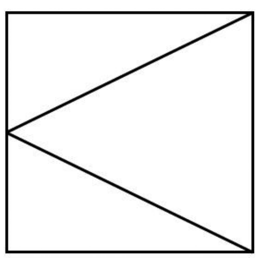

Figure 6.2

(Before you read ahead, please try to think why the students call them equal parts. Please note that in Fig 6.1 and 6.2, even though the students did not specify the lengths of the two line segments, they are assuming the line segments in each square to be of equal lengths. Also the triangles formed in Fig 6.1 are isosceles triangles, and the line segments in Fig 6.2 meet each other at the midpoint of the side of the square.)

Yes, in the case of Category VI (a), they think so because they have considered *equal* to be of *equal quantities or equal area*. They are certainly not wrong because the word 'equal' is very ambiguous. This is a great opportunity to talk about a few important points, starting with - What do you mean by 'equal'? Those who consider their answer of Category VI is to be correct say that 'equal' means 'equal quantity'. Others say that 'equal' means 'same shape and same size'. The class arrives at a conclusion that depending on how we define *equal*, we can accept or reject Category VI. We move ahead with the meaning of *equal* as *same shape and same size*[2].

The students who see Category VI (b) as equal parts see them as equal because all the three parts are triangles and they aren't able to see the difference between the shapes of the triangles. It usually takes a little time for those students to realise that the right angled triangles are half the size of the isosceles triangle.

Next, we ask the students to find out whether there could be more than two ways of dividing the square in Category IV. Most of them don't see more ways but there is always one or two in the class who think of this:

Category - VII

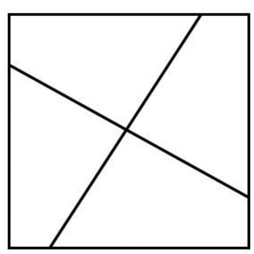

Figure 7.1

To see that all the four parts could be of the same shape and size is not easy for many. Those who might have spent a considerable amount of time in observing shapes and patterns closely while they were younger, are able to spot that the four parts could be identical. For others, they can be asked to try to cut out the shapes (after being instructed how one needs to draw the lines so that the four parts can be equal) and verify. For students in high school, this could be a good question to *prove*. Some may also observe that the lines have to be perpendicular to each other if the four parts have to be equal (Why?[3]). Again, they are asked to find out how many possibilities exist and they come up with the answer - **many many** (**Why?**). Some students also start feeling that the answer is always **many many** and say that there are **many many** ways to divide a square into three equal parts using two lines. While some students object to this, they are unable to reason it out. We will leave it to the reader to explore the question - Prove or disprove that there are infinite ways of dividing a square into four congruent parts using two lines.

Some students draw the picture below also.

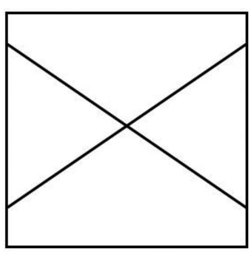

Figure 7.2

In such instances, we need to reiterate the meaning of *equal as same shape and size*. Still, some see the four parts as being of equal shape and size. If there are students who see it that way, they need to be asked to see the four parts and see what kinds of shapes are formed. Such students are usually very few in number but the fact is that they see the four shapes as equal parts which is something that the teacher might not have imagined.

Cutting a square into equal parts using three lines

The next question that the students explore is - *In how many ways can a square be cut into equal parts using three lines?* Most students find out the two ways shown below.

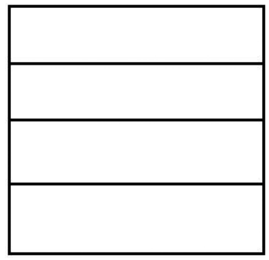

Fig 8.1

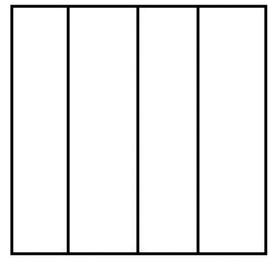

Fig 8.2

Some observe that using three lines, we could divide the square into six equal parts.

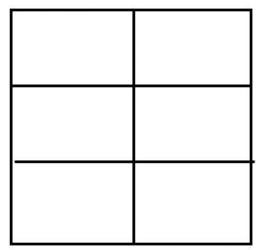

Fig 8.3

By this time, when we ask the students - 'What are the questions that you would like to explore now?' they say, 'How many ways are there to divide a square into six equal parts using three lines and four equal parts using three lines?' Without any further instruction from the teacher, they start pursuing both the problems. Once they come up with their answers, the teacher again asks - 'Why do you feel so?' This is a very important question, both for the teacher and for the student. It is important for the teacher because it is an opportunity to understand how young minds think, and it is important for the students because they start reasoning in response to this question *Why*.

If you happen to try this question in a classroom, there would be more solutions that students might give and many of them would be incorrect. Instead of saying whether it is right or wrong, try asking them *Why do they feel it is correct?*

This exploration can continue and one can ask students to make a table as below and explore by changing the number of lines and the number of equal parts.

Shape	No. of lines	No. of equal parts	How many ways?
Square	1	2	Many many (Why?)
Square	2	3	Two (Why?)
Square	2	4	Many many (Why?)
Square	3	4	Many many (Yes! It is true. But how?)
Square	3	5	How many ways? Why? (Can you prove it?)
Square	3	6	How many ways? Why?
Square	4	8	How many ways? Why?
Square	5	8	How many ways? Why? (There exists a solution)
Square	6	8	How many ways? Why? (The answer is more than 1)
Square	7	8	How many ways? Why?

The next interesting exercise could be to change the shape from a square into some other shape, say a circle or a rectangle or an equilateral triangle and make the same table. Observe if the number of ways will still remain the same for a certain number of lines and equal parts. Can we arrive at some observational generalisations followed by some logical generalisation if possible? Can we try to find (and prove) that for x number of lines in a shape, y number of equal parts are NOT possible? The exploration is endless!

Learning Outcomes

1. Students start observing shapes more closely.
2. A deeper understanding of 'congruence' can be developed before the topic of congruence is introduced in high school geometry.
3. Improving reasoning and observation skills.
4. Attempting observational generalisations and the ability to come up with claims.
5. Importance of *defining* and understanding terms (like *equal*) before engaging in problem-solving.

Questions for further exploration for high school students

1. Try writing down proofs for all the *WHYs*.
2. In a square, we can draw three lines and divide the square into at most seven parts. Is it possible to divide the square into seven parts of equal area? Can we justify our answer?
3. If there are n number of lines drawn, what are the possible number of equal parts that can be made from a square and other shapes?

Teacher Notes

1. I have observed that most students in middle school who use the word 'infinite' confuse the term with 'uncountable' or 'a very large number' or 'the largest number'. Hence, I use a new term with them which is **many many**. In this article, I have used the term **many many** for 'infinite' as the term 'infinite' won't be clear to most middle school students.
2. We don't use the word 'congruence' because it is not a vocabulary that they are familiar with. But 'same shape and same size' is something that early middle school students can comprehend.
3. Wherever there is a 'Why' put up, it is a question for the reader to see if they can prove it with some mathematical rigour.

VINAY NAIR is the co-founder of Raising a Mathematician Foundation. He conducts various online and offline programs in different parts of India on exploratory learning in Mathematics and ancient Indian Mathematics. He aspires to create a research mentality in the minds of school children. He can be reached at vinay@sovm.org.