Addendum to Theorem concerning A MAGIC TRIANGLE

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In the July 2018 issue of *At Right Angles*, the topic of magic triangles was explored, a 'magic triangle' being "an arrangement of the integers from 1 to n on the sides of a triangle with the same number of integers on each side so that the sum of integers on each side is a constant, the 'magic sum' of the triangle." [1] The number of integers on each side is the 'order' of the magic triangle; it is equal to $(n + 3)/3 = n/3 + 1$.

The following result was stated in the article: *The vertex numbers of a fourth-order magic triangle, when arranged in order, form an arithmetic progression*. This is illustrated by the magic triangle in Figure 1, where the vertex numbers are 1,2,3.

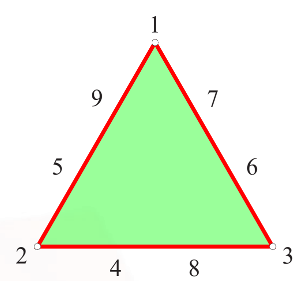

Figure 1. Fourth-order magic triangle with magic sum 17

A few weeks back, this author received an email from Mr James Metz of Hawaii, pointing out that this result is in error. (He also offered more observations on fourth order magic triangles. See his article later in this issue). As proof, he offered a few counterexamples (Figure 2):

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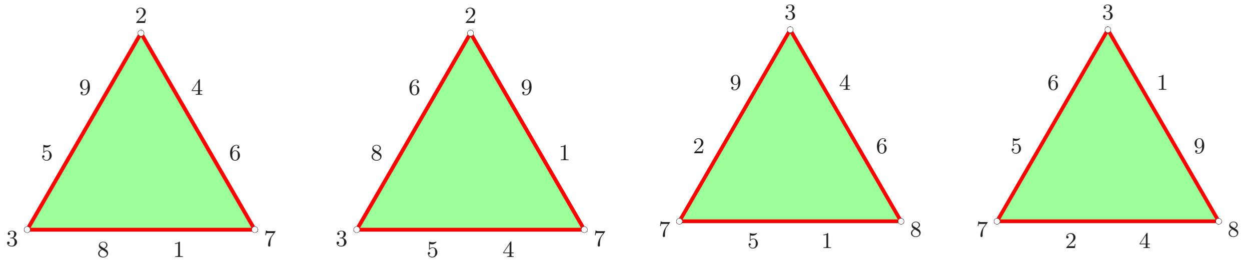


(a) Magic sum 19 (b) Magic sum 19 (c) Magic sum 21 (d) Magic sum 21

Figure 2. Counterexamples to the stated claim

A remarkable state of affairs: we 'proved' the result, yet here we find four different counterexamples to the claim! This challenges us to find out where we went wrong in the supposed proof. This article concerns itself with tracking down the error.

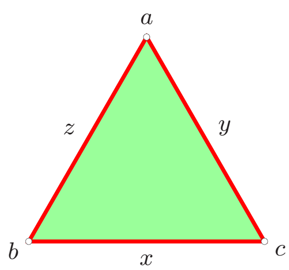
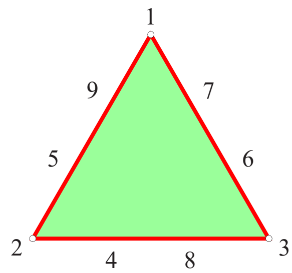


Figure 3. General relationships for fourth-order magic triangles (n = 9)

Proof. Let us recall our 'proof'. We denoted by a, b, c the numbers at the vertices (Figure 3); by x, y, z the sums of the other two numbers on the three edges respectively (x on edge bc; y on edge ca; z on edge ab); and by s the magic sum of the triangle. We deduced that $a + b + c = 3s - 45$ and noted that this means that the sum of the vertex numbers is a multiple of 3. Next we showed that $17 \le s \le 23$. (This follows from $a + b + c \ge 1 + 2 + 3 = 6$ and $a + b + c \le 9 + 8 + 7 = 24$.) We then looked at each possible value of s in turn. We go over these arguments in brief.

s = 17: This possibility implies that $a + b + c = 6$ and takes place if and only if $\{a, b, c\} = \{1, 2, 3\}$. In this case the vertex numbers form an AP, as required. Figure 4 displays one of the magic triangles corresponding to this situation.



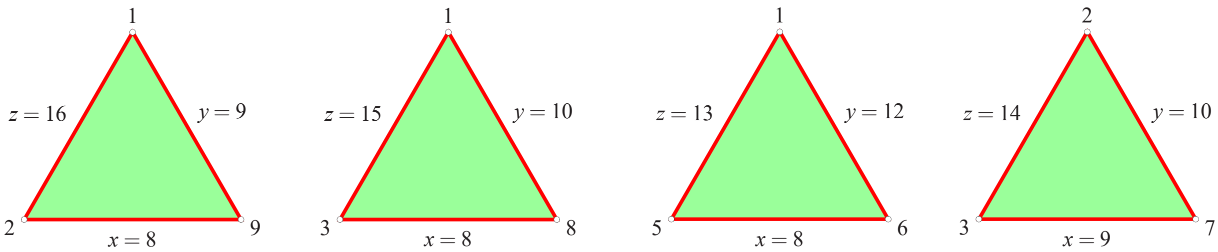
Note that x, y, z are sums of *pairs* of numbers.

Figure 4. Fourth-order magic triangle with magic sum 17

s = 18: This possibility cannot occur. For, if $s = 18$, then $a + b + c = 9$. The sets of three distinct integers between 1 and 9 (inclusive) whose sum is 9 are {1,2,6}, {1,3,5} and {2,3,4}. Consider the first possibility. By focusing on the possible position of 9, we discover that the magic triangle cannot be completed; in each case, some number is required in two different locations, i.e., two copies of that number are required. Hence there is no fourth order magic triangle with vertex numbers 1,2,6 and magic sum 18. Noting the role played by 9, we call it a *witness* to the impossibility of this configuration.

The other possibilities listed also do not work; once again, 9 acts as a witness to show their impossibility. Hence if $s = 18$, the statement that the vertex numbers form an AP is vacuously true.

s = 19: This implies that $a + b + c = 12$. The sets of three distinct integers between 1 and 9 (inclusive) whose sum is 12 are {1, 2, 9}, {1, 3, 8}, {1, 4, 7}, {1, 5, 6}, {2,3,7}, {2, 4, 6} and {3, 4, 5}. The sets that need examination are {1, 2, 9}, {1, 3, 8}, {1, 5, 6} and {2, 3, 7} (in the remaining three cases, the vertex numbers already



(a) Witness: 7 (b) Witness: 7 (c) Witness: 7 (d) Witness: 7 ???

Figure 5. Analysis of fourth-order magic triangles with magic sum 19

form an AP). The first three cases are studied in Figure 5 (a), 5 (b) and 5 (c). As earlier, in each case we need a witness that plays the role earlier played by 9. The relevant witnesses are listed alongside the captions. (Please check that they fulfill their duties faithfully.)

What about the fourth case, depicted in Figure 5 (d)? We had claimed earlier that 7 is again a witness, and we left the missing steps in the argument to be filled in by the reader. *But this is where we went wrong; 7 does not work out as a witness.* Indeed, no witness can be found for this configuration! And as the counter example provided by James Metz shows, there really is a magic triangle with vertex numbers 2, 3, 7 and magic sum 19 (Figures 2 (a) and 2 (b)).

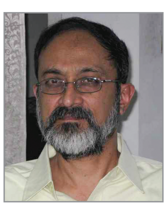
It follows that if s= 19, the claim that the vertex numbers form an AP is not true. We have found the error in our reasoning.

By virtue of the other result proved in the original article (that proof is valid, there is nothing wrong with it!), where we found a mapping between magic triangles with magic sum s and magic sum $40 - s$, we infer that the claim that the vertex numbers form an AP will be false for the case $s = 21$ as well. The counterexamples found by James Metz are consistent with this statement.

Acknowledgement. I thank James Metz most sincerely for writing to me and pointing out this error. It is always a humbling experience to an author when an error is spotted. It shows the extreme need for accuracy in one's reasoning and one's writing. It also shows the extreme need for care in not passing off to the reader the task of checking an argument!

References

[1] Wikipedia. "Magic triangle (mathematics)." https://en.wikipedia.org/wiki/Magic_triangle_(mathematics)



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