The Truncated Icosahedron – An Iconic 3-D Shape

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A polyhedron is a 3-D shape whose boundaries or faces are planar polygons. Two faces meet along an edge while three or more faces meet at a vertex. Among polyhedra we have the five Platonic solids. All the faces of a Platonic solid are congruent regular polygons. These five were identified by ancient Greek geometers who also proved that no other polyhedron has this property (i.e., of having faces that are regular polygons of the same size and the same number of sides). One example of a Platonic solid is the icosahedron, which has 20 faces that are congruent equilateral triangles. It has 30 edges and 12 vertices (Figure 1).

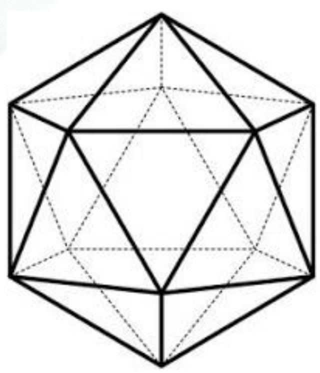

Figure 1

There is another class of semi-regular 3-D shapes - the Archimedean solids. In these the faces are again regular polygons but they are not all congruent, i.e., they differ in

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the number of sides. One such Archimedean solid is the truncated icosahedron. Now look at the icosahedral shape and imagine that all of its 12 vertices are sliced off by identical and symmetrical cuts. The resulting shape is the truncated icosahedron (Figure 2).

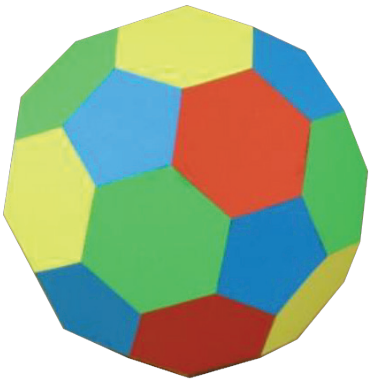

Figure 2

As a result of this action we have the following:

A. The original triangular faces have become hexagons. At every vertex of the original shape there now appears a pentagon. So the new shape has 32 faces – 20 regular hexagons and 12 regular pentagons. No two pentagons touch each other.

B. Each vertex of the original shape is replaced by 5 new vertices, and so the new shape has 60 vertices.

C. Around every vertex of the original shape 5 new additional edges are created. That is, 60 new edges are created in addition to the existing 30, bringing the number of edges of the new shape to 90.

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There are two well-known results that apply to all polyhedra, regular or otherwise.

(a) Euler's relation. This states that the sum of the number of faces and the number of vertices of a polyhedron is 2 more than the number of edges. Clearly this holds for the truncated icosahedron ($60 + 32 = 90 + 2$).

(b) The sum of the angles at the vertices of the polygons that meet at any vertex of the polyhedron is necessarily less than 360°, else it would just flatten out. The shortfall, i.e., the difference between 360° and the sum as mentioned above, is termed the angular defect of the said vertex. The sum of the angular defects of all the vertices of a polyhedron always equals 720°. (This statement can be considered as the 3-D analog of the assertion that the exterior angles of any polygon sum to 360°.) In the case of the truncated icosahedron two hexagons and one pentagon meet at each vertex. The sum of the angles at their vertices is $2 \times 120^\circ + 108^\circ = 348^\circ$. The angular defect at one vertex is thus 12°. Since there are 60 similar vertices the total angular defect is $60 \times 12^\circ = 720^\circ$.

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The truncated icosahedron is the shape of the soccer ball, one of the best known and best loved objects (Figure 3).

Figure 3

The shape was suggested by Eigil Nielsen in 1962. A standard soccer ball is made of vulcanised rubber units stitched together or thermally sealed together and inflated. Typically the pentagons are black in colour while the hexagons are white. An array of conjoined hexagons would just be planar, while an array of pentagons closes abruptly to form a dodecahedron, not a shape that would roll easily. The above combination of both pentagons and hexagons seems to achieve a balance, giving a well-rounded and pleasing shape.

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Among the chemical elements carbon has a unique and special place. This is due to the ability of carbon atoms to form strong bonds with each other enabling the formation of large structures and complex molecules. Carbon is the basis of life itself. The branch of organic chemistry deals with the study of carbon compounds. For a long time pure carbon was known to exist in two forms – graphite and diamond. In the year 1985 a new form of carbon was observed to form in laboratory conditions that mimicked the conditions in red giant stars. Further studies revealed that in this form 60 carbon atoms are bound together in a stable structure - yes, the truncated icosahedron. The carbon atoms occupy the vertices and the C-C bonds form the edges, of which 60 are single bonds and 30 are double bonds. Each carbon atom is linked to two other carbon atoms by single bonds and to one other by a double bond, satisfying its valency of four. The sides of the pentagons are all single bonds. Double bonds radiate from the pentagons, each linking to another pentagon. There are no faces as such, as it is a hollow skeletal arrangement (Figure 4).

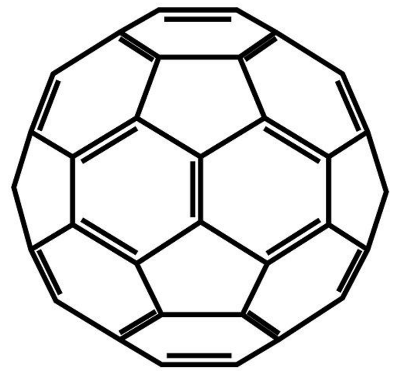

Figure 4

This newly discovered form of carbon was named Buckminsterfullerene, or Fullerene (informally, buckyball) for short, in honour of Buckminster Fuller of the 'Geodesic dome' fame. The structure of this form of carbon reminded the chemists of geodesic domes which are shell-like spherical or hemispherical structures made of triangular light-weight units; they are much stronger than one would expect and can be built to an indefinitely large (or small, as above) scale.

Some micro-organisms have external structures that are slight modifications of the icosahedral shape.

The truncated icosahedron is thus a truly iconic shape that links geometry, sport, chemistry and architecture.

Acknowledgments

Figures 1-4 are taken from the following websites:

https://shutterstock.com

https://www.korthalsaltes.com/model.php?name_en=truncated%20icosahedron

https://pt.wikipedia.org/wiki/Ficheiro:Soccer_ball_animated.svg

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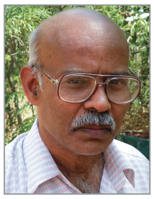

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