TechSpace

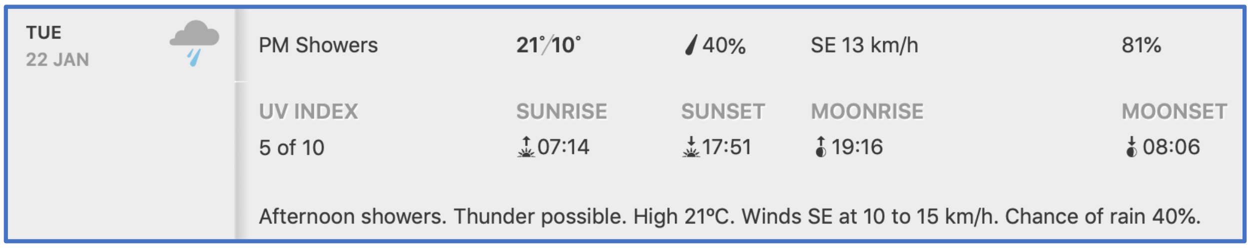
Taking a Chance with a Graphics Calculator

BARRY KISSANE

Graphics calculators have been available to students in secondary school in some countries now for more than thirty years, although of course their capabilities have been developed in various ways to support the school curriculum over that time. The most frequent use of these devices seems to be concerned with the representation of functions, including in particular their graphical representation, which was an important component of a previous paper in this magazine (Kissane, 2016). However, the success of graphics calculators is due in no small part to their use for a much wider range of mathematical capabilities. In this article, the focus is on their potential to help students to learn about chance phenomena, which are generally addressed in schools through the study of probability.

The history of probability in secondary schools is relatively short and generally unfortunate. Unlike many other parts of the secondary school curriculum, such as algebra, geometry, trigonometry and calculus, probability has been studied in schools only recently, and was relatively rare in most countries as little as fifty years ago. One part of the reason for this is likely to be that probability is a relatively recent inclusion in mathematics itself, dating from around the sixteenth century (Hacking, 1975). Until quite recently, much of the probability work in schools has been excessively formal, with a focus on the algebra of probabilities, but with less attention paid to the nature of everyday random phenomena. Yet in recent times, probabilities have become more evident and explicit in our daily world, a good example of which is weather forecasting, now regularly accessed by many people on their smartphones.

Keywords: probability, chance, fair, forecasting, dice, coin, card, Bernoulli event, binomial distribution, technology, graphic calculator



Weather forecast for New Delhi from Weather.com

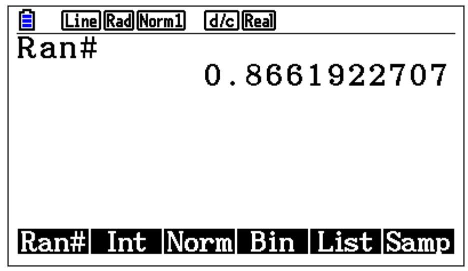
To give an example, the screenshot above is taken from a popular online weather forecasting website (https://weather.com), suggesting that the chance of rain on a certain recent day in New Delhi, India was 40%.

An important part of learning about probability is to understand what such statements mean, and a graphics calculator can be of value for this purpose. A key intention of this article is to explore some of the possibilities now available.

Random numbers on calculators

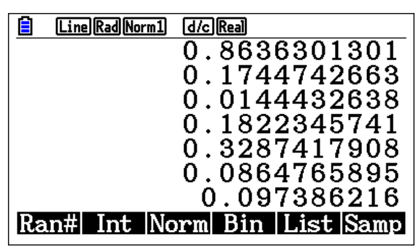
Kissane and Kemp (2014a) claimed that calculators could be of value to education beyond their obvious role to facilitate and undertake computations. Calculators could help develop understanding of mathematical concepts, could allow students to undertake personal explorations in mathematics, and offer opportunities for their hypotheses to be confirmed or to be contradicted, either of which is helpful for learning. In this article, examples of all of these will be offered, using in particular a recent graphics calculator, the CASIO fx-CG50, to illustrate these. Some of the ideas presented are elaborated in more detail in Kissane and Kemp (2014b, Module 7).

The essential ingredient of opportunities to explore chance on calculators is a random number generator, allowing a user to generate a random number between zero and one with a single key press. This capability is present on all graphics calculators and almost all recent scientific calculators. A single press of the relevant command on the CASIO fx-CG50 is shown below.

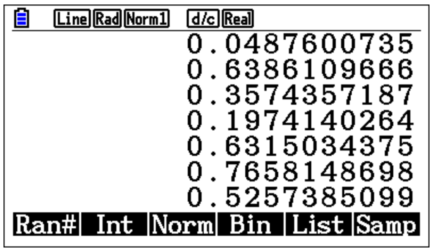


The number generated is not, of course, actually random. It is generated by the calculator, a predictable device, in the form of a *pseudo-random* number, requiring a sophisticated internal algorithm. Importantly, numbers of this kind behave in similar ways to random numbers and thus can be used to simulate and to study random phenomena. On this calculator – and on other calculators – the random numbers are generated with a uniform distribution on the open interval (0,1).

To begin to explore how random phenomena work, you can generate a succession of random numbers, or each of a group of people – such as a class – can each generate a random number and see what happens in the shorter and longer terms. Here is an example of generating seven random numbers in succession (all that will fit on a single screen of this particular calculator):



This screen allows a useful observation to be made about random phenomena: they are much easier to predict in the long run than in the short run. While in theory, for example, half the numbers generated should be larger than 0.5, and half less than 0.5, this is not expected to be evident in such a small collection of observations. In this case, only one of the seven numbers shown is larger than 0.5. Further experimentation will show that other results will occur, such as the following screen, in which four of the seven numbers exceed 0.5.

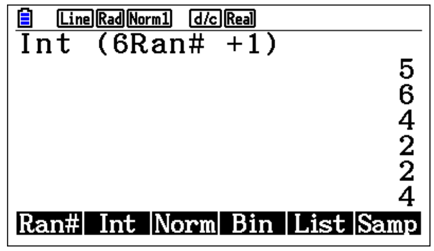


It is rare that random numbers are useful in their basic form, as a number in (0,1). So calculators often include pre-programmed ways of transforming them for various purposes. A common example involves the production of random integers, such as random integers from 1 to 6, to simulate rolling a fair six-sided die. As well as dedicated random commands (which are not used here) the same effect can be achieved with a transformation using the Integer function (Int) in order to obtain the integer part of a number. The table below summarizes this approach:

Command	Random number result
Ran#	between 0 and 1
6Ran#	between 0 and 6
6Ran# + 1	between 1 and 7
Int(6Ran# + 1)	integer from 1 to 6

This kind of transformation is so fundamental to work with simulation that it might reasonably be argued to be an essential part of any modern curriculum in probability, in fact, in an age when technology is often available. Understanding such transformations is a key pre-requisite for designing Monte Carlo simulations, which have become prominent since the age of the computer.

On the calculator, the last of the commands above is used below (six successive times) to produce a set of six simulated dice rolls for a standard die:



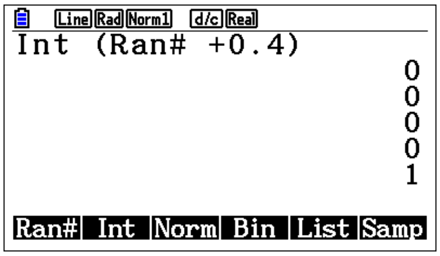
Once again, the essential unpredictability of random phenomena is shown here. It is much quicker and easier (and also quieter) for someone to generate dice rolls in this way on their calculator to study what happens than it is to use actual dice. In this case, each tap of the Execute key on the calculator generates and records another dice roll. When students study probability, and learn to compute probabilities of various results (such as a probability of 0.5 of obtaining an even number on a single roll of a fair die), their understanding is enriched by opportunities to see that this does not mean that an even number will be obtained 50% of the time, with only a few rolls. The result is instead a long-run expectation; such is the intrinsic nature of random events. Expecting long-run patterns to be evident in the short run is perhaps the most common problem people have with random events.

Understanding weather forecasts

Rather than integers, some phenomena are well modelled as Bernoulli events, for which the result is one of two possibilities, usually referred to as 'success' or 'failure'. The weather forecast shown earlier is a good example. The website predicts that it will rain on a particular day with a probability of 40%. That is, they predict that rain will be a 'success' represented by 1, 40% of the time and a 'failure' represented by 0, on the remaining 60% of the time. To understand such predictions, it is helpful to simulate a succession of days of that kind on the calculator. The appropriate command is Int(Ran# + 0.4), which will have the value of 1 (i.e. it rains) on 40%

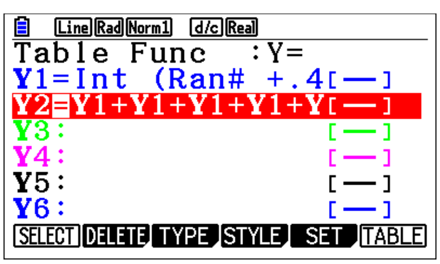
of occasions and 0 (i.e. it does not rain) on the other 60% of occasions.

Here is a simulation of five days in succession, using this command, assuming that the chance of rain on each day is 40%:

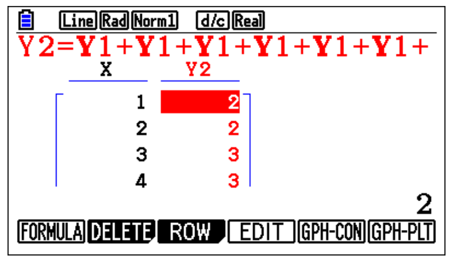


Some students – and some citizens – might regard these forecasts as defective, when there are four days in succession without rain. Instead, they need to recognize that the nature of random phenomena is such that a result of this kind is not especially unlikely: the probability of 40% applies to the long run, but not necessarily to the short run.

These kinds of capabilities, which are available for many scientific calculators, allow students to experience randomness for themselves. Because of its larger screen and other capabilities, a graphics calculator allows more substantial simulations to be undertaken, however, with a better opportunity to see what happens with a relatively large number of events. For example, the CASIO fx-CG50 allows students to generate a sequence of Bernoulli events, and then to add them to produce in effect elements from a simulated binomial distribution, which can then be studied as data. The screen below shows how a single simulation is stored in variable Y1 and then seven successive (different and independent) simulations of that kind accumulated in Y2 to show the number of rainy days in a week, when each of the days independently has a 40% chance of rain.

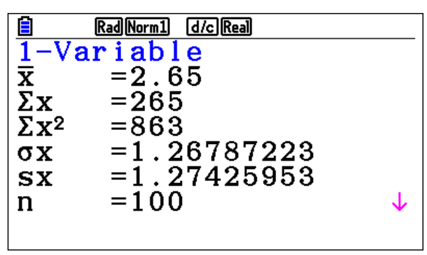


The result of such a simulation in Y2 effectively represents observations from a binomial distribution for which the probability of success is 0.4 and there are seven repetitions. While students might (and should) study the binomial distribution theoretically, there is value in first seeing its origins in this way and examining the consequences of repeated random observations of this kind. On the calculator, when used in this way, results are provided in a table, which can be scrolled easily to see the variation of results. The screen below shows one such simulation of 100 'weeks':



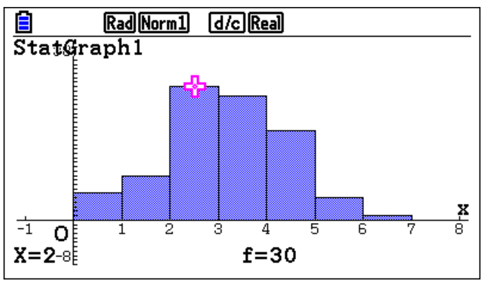
While intuition might expect a 'typical' week to have 7 x 0.4 or somewhere between two or three rainy days per week, of course in reality there are variations, evident from scrolling the table, only the first four elements of which are shown here.

It is difficult to compare tables of 100 elements, however. The capacity of this calculator to readily analyse the (finite) table as a data set and not a function table overcomes this problem. (Some other graphics calculators do not permit this alternative, because tables are not represented as finite lists.) Analyses might take any of several forms. For instance, a numerical analysis shows that, in the longer term (in this case with 100 observations), the mean number of rainy days in a week was 2.65, with a standard deviation of 1.27. At a later point, students might encounter the theoretical mean and standard deviation

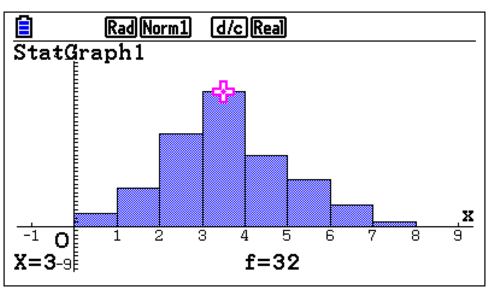


of this binomial distribution (2.8 and 1.30 respectively), but the simulation results give a sense of what might happen in practice, before such theoretical analyses are available.

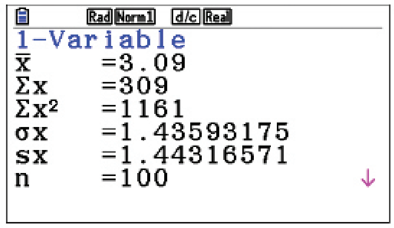
Graphical comparisons can be more evocative than numerical analyses, of course. The calculator routinely provides these as well. In this case, the histogram below gives a sense of what happened in the 100 simulated weeks. Scrolling the histogram shows that there were 30 weeks with two rainy days, slightly more than the number of weeks (28) with three rainy days. However, the graph also shows that there were six weeks with no rainy days at all, one week with six rainy days and no weeks at all for which it rained on all seven days. Such is the nature of randomness.



Importantly, each time the simulation is conducted in this way by someone, a different result is generated; so that one person can undertake experiments like this repeatedly to get a feel for the outcomes and their typical variation. In a classroom, each student will have a different table from every other student, providing a rich opportunity to see what is typical and consistent about a situation that is ultimately random. To illustrate this variation, the screen below shows the results of a second simulation of 100 weeks, conducted in the same way, and using again the same calculator settings.



As each day is simulated at random, the difference between the two simulations is entirely due to the randomness involved. The graph of the second simulation shows both similarities and differences from that of the first. This time, there are more weeks with three wet days than two wet days and there was even a week for which it rained every day. The numerical summary also shows some differences:

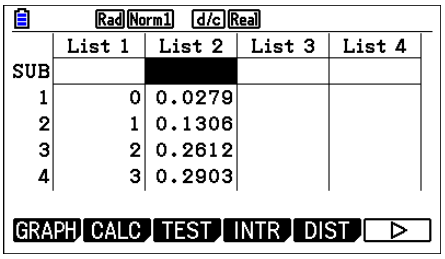


The mean number of wet days (3.09) is larger than previously, and the standard deviation (1.44) is also larger than previously. However, the overall shape of the distributions is similar – peaked in the middle with tails on each end, and with a similar slight skew. While different from each other, the numerical statistics remain close to the long-term theoretical values.

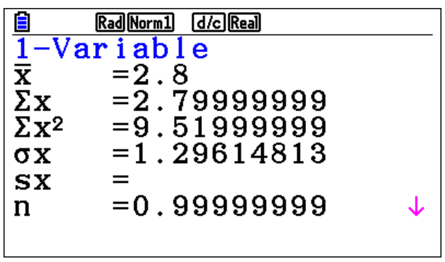
In addition, the situation can be studied with a larger number of 'weeks', in order to appreciate what happens on the longer term. In effect, the calculator is a personal experimental device.

These sorts of experiences – readily repeated on the calculator – provide opportunities to see both short-term and long-term behavior, and to appreciate the difference between the theoretical expectations and their likely practical consequences. They also offer students an opportunity to see for themselves that, even though the events simulated are random, there is a remarkable consistency of results in the longer term – much less visible in the shorter term – which is what makes the formal study of probability valuable, of course.

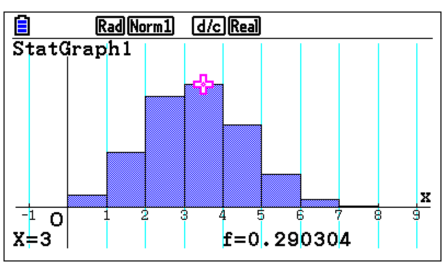
Explorations of these kinds are perhaps most appropriate before theoretical analyses are undertaken, in order to build intuitions about random phenomena, including an expectation for both short-term variation and longer-term stability. However, at a later stage, after theoretical studies have been undertaken, it is also valuable to use the calculator to generate and to show the results. This too is readily done with the CASIO fx-CG50, as shown in the next three screens. In the first screen, the first few terms of the binomial probability distribution for n = 7 and p = 0.4 are shown.



A numerical summary of this distribution is available, showing (within rounding errors) the theoretical mean of 7 x 0.4 = 2.8 and the theoretical standard deviation of 1.30.



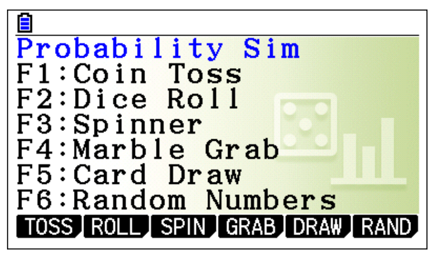
A graphical representation of this distribution shows the characteristic binomial distribution shape, with the vertical axis now showing theoretical probabilities, rather than simulated frequencies:



To understand and analyze situations which involve randomness, and to make predictions about likely outcomes, a theoretical probability model is very important. However, it is also important to build an understanding of the fact that it is a theoretical model, explaining long-term aggregated behavior, and of less practical significance for dealing with the short-term behavior in which we are often interested, such as whether or not it will rain tomorrow, or next week, once we are advised that the probability of rain on any day is 40%. Simulations on a graphics calculator are perhaps even more helpful to build this kind of understanding than are the theoretical models.

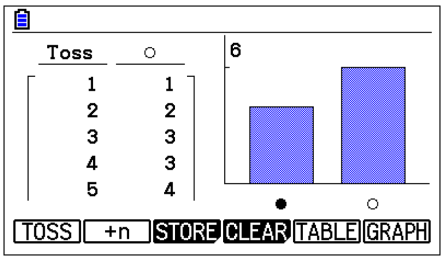
A calculator application for simulation

The various calculator explorations described here make use of the standard features of a graphics calculator like the CASIO fx-CG50, including the generation of random numbers, the tabulation of functions and the analysis of statistical data. However, simulations are so helpful for understanding chance phenomena that it is not surprising that the calculator also includes a separate application that is devoted to this area. The *ProbSim* application on the calculator supports various kinds of probability simulations, as suggested by the opening screen shown below:

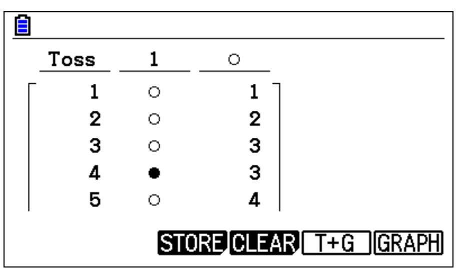


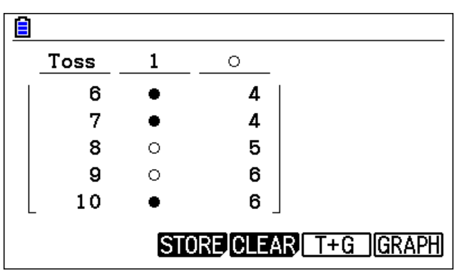
The titles of the various kinds of simulations offered in this application are reminiscent of the typical scenarios discussed in elementary probability studies … tossing coins, rolling dice, playing cards, taking marbles from urns, and so on. However, these can be used to simulate various random phenomena that are consistent with models of those kinds, as well as the actual situations described. The advantages for users of the calculator are that the various simulations are relatively easy to configure in this environment, a large number of results can be obtained fairly quickly and they can be seen in various ways.

As an example of a benefit of this application, consider the analysis of runs of random results. David Moore (1990, p.120) observed that people often intuitively underestimate the probability of runs in random sequences. So, when asked to write down a sequence of heads and tails imitating 10 successive tosses of a fair coin, he suggested that most people write a sequence with no runs of more than two consecutive heads or tails, consistent with this defective intuition. On the calculator, a set of ten successive coin tosses is readily simulated and the results are then available for scrutiny. A summary of one simulation is shown below:



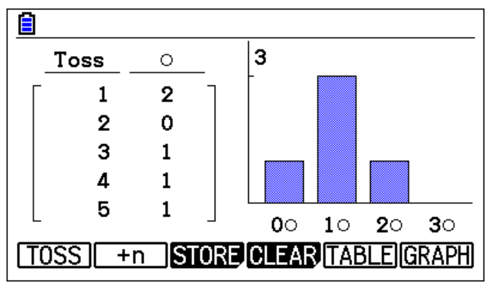
In the screen, 'tails' is represented with a black circle, while 'heads' is represented as a blank circle. Each toss has been recorded, and the screen above shows a graphical summary of the outcome, with four tails and six heads. The table shows the cumulative number of heads after various numbers of tosses. A more thorough investigation of runs is available by choosing to show results in tables, however, as the next two screens illustrate:



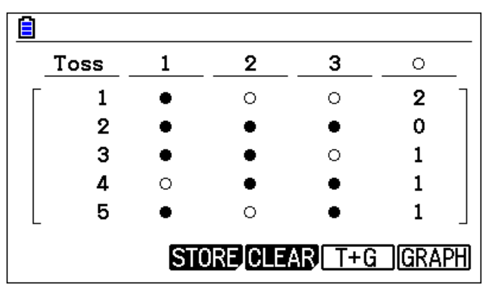


In this case, the tables show clearly that there was a run of three heads (in the first three tosses). It is easy to repeat simulations of these kinds, or for a group of students to independently conduct a simulation, to give a sense of what is 'typical'. At a later stage, such phenomena can be theoretically analysed, but that is too difficult for introductory studies in this area. As Moore notes, the probability of a run of three or more heads is a little over 0.5, so is much less unusual than people think intuitively; the probability of a run of at least three heads or at least three tails in ten coin tosses is even larger – greater than 0.8 – and thus is much more likely to happen than to not happen (1990, p.121).

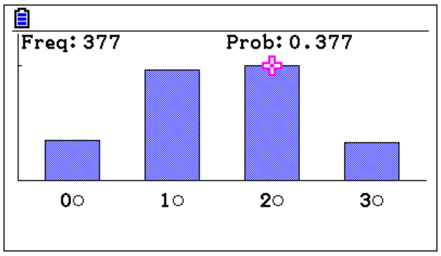
In a similar way, when families are being studied, under an assumption that a newborn baby is equally likely to be a boy or a girl, simulation is a useful tool to examine what possibilities are involved. Thus, the screen below shows a suitable simulation, with boys being represented by dark circles and girls by clear circles. Again, the initial screen shows the number of girls in each of the (five) simulated families:



Alternative screens show the same data differently, however. The screen below shows that even though three simulated families have only one girl, the birth orders in each case are different.

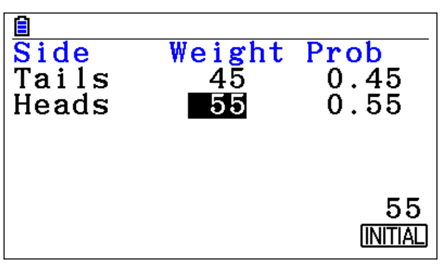


Once again, the graphics calculator is an ideal tool to explore both short-term experiments (like those above) and long-term experiments. For example, the numbers of girls in families is more clearly symmetrical and indeed consistent with theoretical expectations if a large number of families is simulated. The screen below shows one set of results after a thousand three-child families has been simulated.

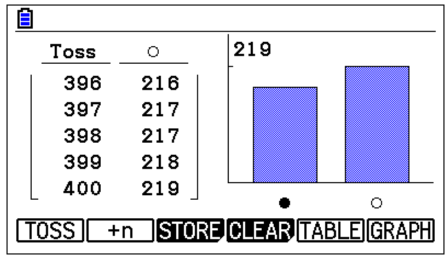


As might be expected, the proportions of families with various numbers of girls after such a large number of trials is close to 12.5%, 37.5%, 37.5% and 12.5% – the theoretical values. But also as expected, the proportions do not exactly equal the theoretical values. Both of these kinds of observations are helpful for learning about randomness.

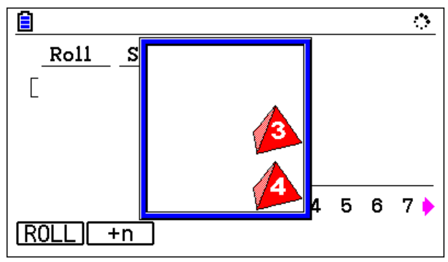
The *ProbSim* application allows for several other kinds of simulations, as noted above. Space precludes exploring these in fine detail, but some observations about the range of possibilities are appropriate. Although it is possible to conduct simulations with everyday objects, such as actual dice, coins and spinners, it is generally more difficult to do so with processes that are not equally likely. In the case of the calculator, adjustments can be made for this purpose. An example involves tossing an unbalanced coin that lands heads 55% of the time. On the calculator, parameters can be set for this sort of purpose, as shown below:



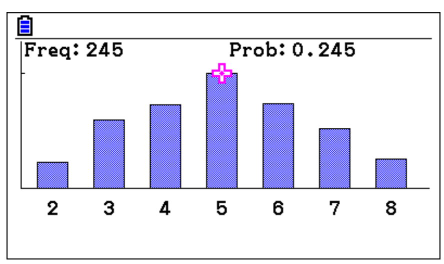
The resulting simulations show that, while a bias of this kind might not at first be clear, it becomes evident after many tosses. In the screen below, for example, the preponderance of heads is clear after the relatively small number of 400 tosses.



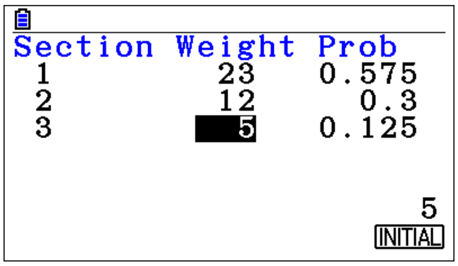
In addition, when students experiment with dice, they are generally restricted to fair six-sided dice, as these are generally the only ones available. However, in the *ProbSim* application, other alternatives are available; the screen below shows the use of a pair of fair tetrahedral (four-sided) dice, one of several choices available.



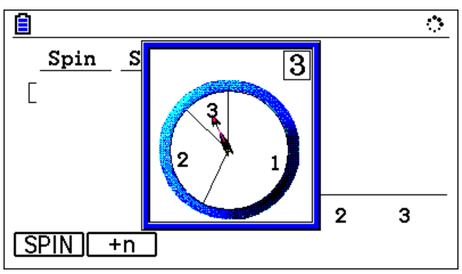
The long-term result of a simulation, after 1000 tosses of these two dice, produces a symmetric distribution, similar to that for a pair of six-sided dice that might normally be accessible. Just as the most likely total for a pair of regular six-sided dice is 7, the most likely total for a pair of four-sided dice is 5, leaving opportunities for students to explain these phenomena.



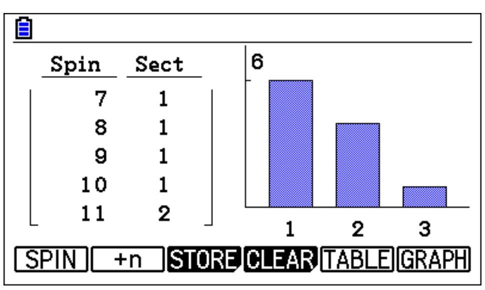
A spinner is another kind of simulation device that is sometimes available in children's games and in classrooms. However, spinners usually comprise a series of equal slices, in order to model equal likelihood. Again, on the graphics calculator application, more flexibility is involved, allowing different probabilities to be modelled easily. Thus, the settings below show a spinner appropriate to modelling the selection of students from a class in which students practice various religions: 23 Hindu, 12 Muslim and 5 Christian.



With an uneven distribution of students into different classes, a spinner that matches the distribution is needed for simulation purposes. As shown below, the calculator automatically uses such a spinner … which under normal circumstances would be hard to accomplish in practice.

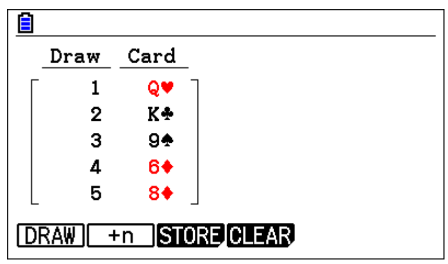


When eleven students are chosen from this class to form a cricket team, the results are generally skewed towards those in the larger groups, as might be expected. A single example is shown below:

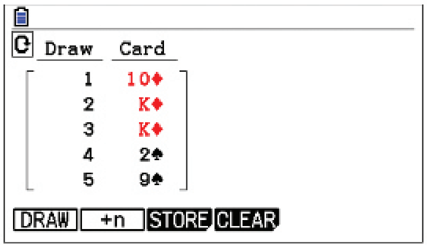


With facilities of this kind, students can learn intuitively that random samples might be expected to be similar to their parent populations, although students will also learn from the same source that random samples can also be expected to produce divergent results as well, especially in the short-term, perhaps helping them to understand notions of 'fairness' in such situations.

Finally, the mathematics of probability was not developed in earnest until mathematicians and others became interested in games of chance, including card games. From that interest developed the much more respectable activity of insurance, without which the modern world could not have existed, according to Bernstein (1998). So it is not surprising that the *ProbSim* application also allows users to experiment with regular playing cards in various ways; although card games are often included in probability texts, they are less often included in experimental work, for practical reasons. The next screen shows a five-card poker hand drawn at random from a regular deck of 52 cards.



Again, students can draw successive hands of cards themselves, to explore how often particular events (such as a pair or a flush) happen, and can also compare their observations with those of others. Importantly for this (and other) applications, the calculator permits sampling to be done either with or without replacement, also a feature of probability theory. In addition, students can choose to have a single pack of cards (which will be exhausted after several hands, if sampling is done without replacement) or several packs of cards (as is routine practice in some professional gambling casinos). Should students elect for sampling with replacement, or for several packs of cards, there is of course a risk that the same hand might contain two cards that are identical, as the example below shows:



The availability of simulation of card games on calculators of course is not intended to encourage gambling; instead, it offers an opportunity for the mathematics of gambling to be studied, and thus for past links between games of chance and the mathematics of probability to be addressed. Indeed, it has been suggested by some organisations such as the Tasmanian Government (2019) concerned with reducing both the prevalence and the problematic impact of gambling that a better understanding of mathematics, instead of an uninformed reliance on intuition, is part of a suitable way of dealing with such problems.

Conclusion

In this paper, we have explored some of the opportunities now available to users of graphics calculators, and especially the CASIO fx-CG50 calculator, to undertake experiments related to probability. The main point is that an intuitive understanding of some features of randomness can be developed using simulations on a calculator. Such an understanding is different from – and even complementary to – the formal development of the mathematics of probability. As suggested by Kissane and Kemp (2014a), a modern graphics calculator offers learning opportunities of different kinds, in addition to computation; these have been illustrated throughout the paper. A conceptual understanding of probability as a long-term limit can be supported in various ways. Students can explore ideas for themselves and both design and execute their own experiments. There are many opportunities for students to predict what might happen in this work, some of them readily confirmed and others contradicted – both of which are helpful for learning. Finally, the basic features of graphics calculators might be used productively for this work, but calculator features designed for simulation purposes offer further power and flexibility.

References

1. Bernstein, P. L. (1998) *Against the gods: The remarkable story of risk*. New York: Wiley.
2. Hacking, I. (1975) *The emergence of probability: A philosophical study of early ideas about probability, induction and statistical inference.* London: Cambridge University Press.
3. Kissane, B. (2016) Introducing differential calculus on a graphics calculator. *At Right Angles*. 5(2), 58-76.
4. Kissane, B. & Kemp, M. (2014) A model for the educational role of calculators. In W.-C. Yang, M. Majewski, T. de Alwis & W. Wahyudi (eds.) *Proceedings of the 19th Asian Technology Conference on Mathematics* (pp 211-220) Yogyakarta: ATCM. (Available for download at http://researchrepository.murdoch.edu.au/24816/) (a)
5. Kissane, B. & Kemp, M. (2014) *Learning mathematics with graphics calculators*. Tokyo: CASIO Computer Company. (Available for free download at http://researchrepository.murdoch.edu.au/24814/) (b)
6. Moore, D. S. (1990) Uncertainty. In L. A. Steen (ed.) *On the Shoulders of Giants: New Approaches to Numeracy*. (pp 95 -138) Washington, DC: National Academy Press.
7. Tasmanian Government (2019) *Know your odds: The longer you play, the more you lose*. Retrieved 21 January 2019 from https://knowyourodds.net.au.

BARRY KISSANE is an Emeritus Associate Professor at Murdoch University in Perth, Western Australia. He has worked with teachers and students to make effective use of calculators for school mathematics education in several countries. In a career spanning more than forty-five years, he has worked as a mathematics teacher and mathematics teacher educator, publishing several books and many papers concerned with the use of calculators. He has held various offices, such as President of the Australian Association of Mathematics Teachers, editor of The Australian Mathematics Teacher and Dean of the School of Education at Murdoch University. He may be contacted at b.kissane@murdoch.edu.au.