Student Corner – Featuring articles written by students.

Pythagorean Triples and Composition

BODHIDEEP JOARDAR

In this discussion I want to demonstrate that:

- If (a_1, b_1, c_1) and (a_2, b_2, c_2) are two Pythagorean triples, then they can be composed to generate the following 6 distinct triples:
    1. $[a_1a_2, (b_1c_2 + c_1b_2), (c_1c_2 + b_1b_2)]$
    2. $[b_1a_2, (a_1c_2 + c_1b_2), (c_1c_2 + a_1b_2)]$
    3. $[(a_1c_2 + c_1a_2), b_1b_2, (c_1c_2 + a_1a_2)]$
    4. $[(b_1c_2 + c_1a_2), a_1b_2, (c_1c_2 + b_1a_2)]$
    5. $[(a_1a_2 - b_1b_2), (a_1b_2 + b_1a_2), c_1c_2]$
    6. $[(a_1a_2 + b_1b_2), (b_1a_2 - a_1b_2), c_1c_2]$
- By such compositions, we can generate infinitely many triples.

We know that Pythagorean triples are infinite in number, and the most common formula for generating triples is to take two relatively prime odd numbers s and t, where $s > t \geq 1$, and produce the triple $(st, \frac{s^2-t^2}{2}, \frac{s^2+t^2}{2})$. *However, can we generate all possible triples from just one triple? Can we generate infinitely many triples from just one triple?* These might be questions worth investigating.

To do this, I take the cue from Brahmagupta's method of composition or *Bhāvanā* as applied to his famous equation *Vargaprakriti* $(x^2 - Ny^2 = 1)$. The method, as we know, can generate all possible solutions from a single solution. So, if I can show that from a known, finite set of roots satisfying the

Keywords: Pythagorean triplets, prime, odd, Vargaprakriti

Pythagorean equation $x^2 + y^2 = z^2$, it is possible to generate other sets of roots, I shall be able to say that the equation has infinitely many roots (see the following section).

Generating triples

For convenience, I begin with two Pythagorean triples (a_1, b_1, c_1) and (a_2, b_2, c_2), instead of one. After developing the formulation, when I know how things are going to develop, I can return to the case of just one triple and proceed from it to generate others.

Composition 1: As the two triples are Pythagorean, they satisfy the equations $a_1^2 + b_1^2 = c_1^2$ and $a_2^2 + b_2^2 = c_2^2$. Rearranging and multiplying the equations, I have,

$$\begin{aligned} a_1^2 a_2^2 &= (c_1^2 - b_1^2)(c_2^2 - b_2^2) = (c_1 + b_1)(c_1 - b_1)(c_2 + b_2)(c_2 - b_2) \\ &= (c_1 + b_1)(c_2 + b_2)(c_1 - b_1)(c_2 - b_2) \\ &= (c_1c_2 + c_1b_2 + b_1c_2 + b_1b_2)(c_1c_2 - c_1b_2 - b_1c_2 + b_1b_2) \\ &= (c_1c_2 + b_1b_2)^2 - (b_1c_2 + c_1b_2)^2 \end{aligned}$$

Therefore, $(a_1a_2)^2 + (b_1c_2 + c_1b_2)^2 = (c_1c_2 + b_1b_2)^2$.

That is, I have a new Pythagorean triple, $(\boldsymbol{a_1a_2}, \boldsymbol{b_1c_2} + \boldsymbol{c_1b_2}, \boldsymbol{c_1c_2} + \boldsymbol{b_1b_2})$.

The way I have obtained this new triple is similar to Brahmagupta's *Bhāvanā* (composition) applied to two given triples. If the triples (a_1, b_1, c_1) and (a_2, b_2, c_2) are named t_1 and t_2 respectively, and the resultant triple $(a_1a_2, b_1c_2 + c_1b_2, c_1c_2 + b_1b_2)$ is named T, then t_1 and t_2 composed individually with T will yield two more triples, one from each composition. Using $\odot$ as the symbol for composition, we may represent the situation as below:

$t_1 \odot t_2 \Rightarrow$ T(new triple); $t_1 \odot$ T $\Rightarrow$ another new triple; $t_2 \odot$ T $\Rightarrow$ yet another new triple.

This is enough to indicate that, with this process continued, infinitely many triples will be generated. In other words, Pythagorean triples are infinite in number. Among the generated triples, there may be 'recurrences' under certain conditions which may or may not exist depending on our choice of operations *(see Properties of Composition)*. However, at no point over an infinite range of compositions will the recurrent triples put an end to the never-ending process of generating new triples *(see the section Infinitely Many Triples and Infinite Recurrences)*.

Obviously, if I start with just one triple (a_1, b_1, c_1), composition can well be applied on itself by the above formula so that $(a_1, b_1, c_1) \odot (a_1, b_1, c_1) = (a_1^2, 2b_1c_1, c_1^2 + b_1^2)$, a new triple. And thus again, infinitely many triples can be generated.

Now to continue with the search for triples.

Composition 2: By interchanging the positions of a_1 and b_1 in Composition 1, that is, by composing $(b_1, a_1, c_1) \odot (a_2, b_2, c_2)$ in the same way as above, I get the triple $(\boldsymbol{b_1a_2}, \boldsymbol{a_1c_2} + \boldsymbol{c_1b_2}, \boldsymbol{c_1c_2} + \boldsymbol{a_1b_2})$.

Composition 3: Similarly, by multiplying the equations as $b_1^2 = c_1^2 - a_1^2$ and $b_2^2 = c_2^2 - a_2^2$, I get the triple $(\boldsymbol{a_1c_2} + \boldsymbol{c_1a_2}, \boldsymbol{b_1b_2}, \boldsymbol{c_1c_2} + \boldsymbol{a_1a_2})$.

Composition 4: Again, by interchanging a_1 and b_1 in Composition 3, that is, by composing $(b_1, a_1, c_1) \odot (a_2, b_2, c_2)$ in the same way, I get the triple $(\boldsymbol{b_1c_2} + \boldsymbol{c_1a_2}, \boldsymbol{a_1b_2}, \boldsymbol{c_1c_2} + \boldsymbol{b_1a_2})$.

Composition 5: Now I can proceed to find the triple with the term c_1c_2. Taking the equations $a_1^2+b_1^2=c_1^2$ and $a_2^2+b_2^2=c_2^2$ as they are and multiplying them, I get,

$$\begin{aligned}(c_1c_2)^2 &= (a_1^2+b_1^2)(a_2^2+b_2^2)\\ &= (a_1+ib_1)(a_1-ib_1)(a_2+ib_2)(a_2-ib_2) \quad \text{[complex factorization; here i} = \sqrt{-1}\text{]}\\ &= (a_1+ib_1)(a_2+ib_2)(a_1-ib_1)(a_2-ib_2)\\ &= [(a_1a_2-b_1b_2)+i(a_1b_2+b_1a_2)][(a_1a_2-b_1b_2)-i(a_1b_2+b_1a_2)]\\ &= (a_1a_2-b_1b_2)^2+(a_1b_2+b_1a_2)^2;\end{aligned}$$

that is, another triple $(\boldsymbol{a_1a_2-b_1b_2, a_1b_2+b_1a_2, c_1c_2})$. It does not matter if $a_1a_2<b_1b_2$; what matters in a triple is the absolute value $|a_1a_2-b_1b_2|$.

Composition 6: Now, as $(a_1a_2-b_1b_2)^2+(a_1b_2+b_1a_2)^2=(a_1a_2+b_1b_2)^2+(b_1a_2-a_1b_2)^2$, there may be yet another triple, $(\boldsymbol{a_1a_2+b_1b_2, b_1a_2-a_1b_2, c_1c_2})$. Again, to avoid negative values, I can take $|b_1a_2-a_1b_2|$. It may be noticed that this is the same as composing $(b_1,a_1,c_1)\odot(a_2,b_2,c_2)$ according to the formula of Composition 5.

Thus, from the two triples (a_1,b_1,c_1) and (a_2,b_2,c_2) [or t_1 and t_2], I have generated by compositions $(C_1,C_2,C_3,C_4,C_5,C_6)$ similar to *Bhāvanā* the following six distinct triples:

$$\begin{aligned}\mathbf{C_1} &:[a_1a_2,(b_1c_2+c_1b_2),(c_1c_2+b_1b_2)]\\ \mathbf{C_2} &:[b_1a_2,(a_1c_2+c_1b_2),(c_1c_2+a_1b_2)]\\ \mathbf{C_3} &:[(a_1c_2+c_1a_2),b_1b_2,(c_1c_2+a_1a_2)]\\ \mathbf{C_4} &:[(b_1c_2+c_1a_2),a_1b_2,(c_1c_2+b_1a_2)]\\ \mathbf{C_5} &:[(a_1a_2-b_1b_2),(a_1b_2+b_1a_2),c_1c_2]\\ \mathbf{C_6} &:[(a_1a_2+b_1b_2),(b_1a_2-a_1b_2),c_1c_2]\end{aligned}$$

It should be noted that, as in Brahmagupta's *Bhāvanā, it is the property that $(x^2\pm Ny^2)$ is 'closed under multiplication' that is made use of in these transformations.*

Notation: C_1, C_2, etc., being the six composition formulas for generating triples, the compositions can be represented as "$C_1(t_1\odot t_2)$", meaning "t_1 composed with t_2 according to formula C_1", or "$C_3[t_1\odot C_2(t_1\odot t_2)]$", meaning "the result of t_1 composed with t_2 according to formula C_2 is composed with t_1 according to formula C_3", and so on.

Primitive or Non-Primitive Triples: Obviously, C_1, C_2, etc., can generate Primitive as well as Non-Primitive Triples. I will be interested in primitive triples (PPT) only, and all numerical examples of triples that I use will be reduced to their corresponding PPTs. This will lead to a search for the conditions under which non-PPTs are generated, and also to the big issue of "recurrence of triples in a continuous process of composition."

How triples multiply

The six formulas, if laid out on an Excel sheet, will go on generating triples if input triples are entered. In Table 1 I have done this, starting with $\mathbf{t}=(3,4,5)$, the smallest triple. I have first done the compositions $C_1(t\odot t), C_2(t\odot t), C_3(t\odot t), C_4(t\odot t), C_5(t\odot t), C_6(t\odot t)$, and reduced them to their corresponding PPTs, say, $\mathbf{T_1,T_2,T_3,T_4,T_5,T_6}$ respectively. Then I have proceeded to compose $\mathbf{t}$ with each of these PPTs. *In the whole process, I have reduced every composition to its corresponding PPT (without negative signs).* Note: I could not compose t with T_6 because $T_6=C_6(t\odot t)=(25,0,25)\Rightarrow(1,0,1)$, a 'trivial' triple. Also, in composing t with $T_5=C_5(t\odot t)$ I have not ignored the negative sign in $C_5(t\odot t)=(-7,24,25)$ because, as I will show later, keeping or ignoring the negative signs can have quite different consequences.

t	T	C_1	$T_1 = C_1/\text{gcd}$	C_2	$T_2 = C_2/\text{gcd}$	C_3	$T_3 = C_3/\text{gcd}$	C_4	$T_4 = C_4/\text{gcd}$	C_5	$T_5 = C_5/\text{gcd}$	C_6	$T_6 = C_6/\text{gcd}$
3	3	9	9	12	12	30	15	35	35	−7	7	25	1
4	4	40	40	35	35	16	8	12	12	24	24	0	0
5	5	41	41	37	37	34	17	37	37	25	25	25	1
gcd		1	1	1	1	2	1	1	1	1	1	25	1
3	9	27	27	36	36	168	21	209	209	−133	133	187	187
4	40	364	364	323	323	160	20	120	120	156	156	−84	84
5	41	365	365	325	325	232	29	241	241	205	205	205	205
gcd		1	1	1	1	8	1	1	1	1	1	1	1
3	12	36	36	48	24	171	171	208	208	−104	104	176	176
4	35	323	323	286	143	140	140	105	105	153	153	−57	57
5	37	325	325	290	145	221	221	233	233	185	185	185	185
gcd		1	1	2	1	1	1	1	1	1	1	1	1
3	15	45	5	60	60	126	63	143	143	13	13	77	77
4	8	108	12	91	91	32	16	24	24	84	84	36	36
5	17	117	13	109	109	130	65	145	145	85	85	85	85
gcd		9	1	1	1	2	1	1	1	1	1	1	1
3	35	105	105	140	140	286	143	323	323	57	57	153	153
4	12	208	208	171	171	48	24	36	36	176	176	104	104
5	37	233	233	221	221	290	145	325	325	185	185	185	185
gcd		1	1	1	1	2	1	1	1	1	1	1	1
3	−7	−21	21	−28	28	40	5	65	65	−117	117	75	3
4	24	220	220	195	195	96	12	72	72	44	44	−100	4
5	25	221	221	197	197	104	13	97	97	125	125	125	5
gcd		1	1	1	1	8	1	1	1	1	1	25	1

Table 1. C_1, C_2, etc., are the composition formulas applied on $t \odot T$.
The ‘recurrent’ triples are highlighted in different colours.

The table shows that 36 triples have been generated out of which, except a few that recur (after ignoring negative signs), all the others are distinct (including a ‘trivial’ one). The calculations may be carried out on an Excel sheet. The procedure has been explained in the Appendix of this article.

Properties of Composition

Based on the composition formulas and with reference to the triples generated by composition in *Table 1*, I can study the properties of composition.

1. **PPTs and Non-Primitive PTs.** Both PPTs and non-primitive PTs (i.e., Pythagorean triples that are not primitive, which means that the three numbers in the triple have a common factor exceeding 1) are generated by the compositions. Thus I had (9, 40, 41), (35, 12, 37), etc., as well as (30, 16, 34), (40, 96, 104), etc. As PPTs are more fundamental, the non-primitive PTs have all been reduced to their corresponding PPTs in this study (with negative signs ignored).

2. ***b* = *c* − *1* and *b* < *c* − *1***. PPTs (a, b, c) being of two kinds, one where the even member $b = c - 1$ and the other where $b < c - 1$, it is found that both kinds are generated by the compositions. So I had (5, 12, 13), (13, 84, 85), etc., as well as (143, 24, 145), (57, 176, 185), etc.

3. **Reversal of order: first two terms.** It is interesting to see what happens when the order of the first two terms is reversed in any one or both of the triples. The composition formulas are so constructed that we have the following consequences (see Table 2 below):

   (1) comparing $(a_1, b_1, c_1) \odot (a_2, b_2, c_2)$ with $(b_1, a_1, c_1) \odot (a_2, b_2, c_2)$: (C_1, C_2) exchange results, (C_3, C_4) exchange results; and (C_5, C_6) not only exchange results, but with the order of the first two terms of the resultant triple reversed;

   (2) comparing $(a_1, b_1, c_1) \odot (a_2, b_2, c_2)$ with $(a_1, b_1, c_1) \odot (b_2, a_2, c_2)$: (C_1, C_4) exchange results with the order of the first two terms of the resultant triple reversed; so do (C_2, C_3) and (C_5, C_6) (ignoring negative signs);

   (3) comparing $(a_1, b_1, c_1) \odot (a_2, b_2, c_2)$ with $(b_1, a_1, c_1) \odot (b_2, a_2, c_2)$: (C_1, C_3) exchange results with the order of the first two terms of the resultant triple reversed; so do (C_2, C_4). But results for (C_5, C_6) remain unchanged if negative signs are ignored.

t_1	t_2	C_1	$T_1 = C_1/gcd$	C_2	$T_2 = C_2/gcd$	C_3	$T_3 = C_3/gcd$	C_4	$T_4 = C_4/gcd$	C_5	$T_5 = C_5/gcd$	C_6	$T_6 = C_6/gcd$
3	35	105	105	140	140	286	143	323	323	57	57	153	153
4	12	208	208	171	171	48	24	36	36	176	176	104	104
5	37	233	233	221	221	290	145	325	325	185	185	185	185
gcd		1	1	1	1	2	1	1	1	1	1	1	1
4	35	140	140	105	105	323	323	286	143	104	104	176	176
3	12	171	171	208	208	36	36	48	24	153	153	57	57
5	37	221	221	233	233	325	325	290	145	185	185	185	185
gcd		1	1	1	1	1	1	2	1	1	1	1	1
3	12	36	36	48	24	171	171	208	208	−104	104	176	176
4	35	323	323	286	143	140	140	105	105	153	153	−57	57
5	37	325	325	290	145	221	221	233	233	185	185	185	185
gcd		1	1	2	1	1	1	1	1	1	1	1	1
4	12	48	24	36	36	208	208	171	171	−57	57	153	153
3	35	286	143	323	323	105	105	140	140	176	176	−104	104
5	37	290	145	325	325	233	233	221	221	185	185	185	185
gcd		2	1	1	1	1	1	1	1	1	1	1	1

Table 2. C_1, C_2, etc. are the composition formulas applied on $t_1 \odot t_2$.

4. **Reversal of sequence of composition: Commutativity check.** In constructing the formulas, I assumed the sequence $t_1 \odot t_2$, where $\mathbf{t_1} = (a_1, b_1, c_1)$, $t_2 = (a_2, b_2, c_2)$. When the sequence is reversed and $t_2 \odot t_1$ is composed it is found that $C_1(t_1 \odot t_2) = C_1(t_2 \odot t_1), C_3(t_1 \odot t_2) = C_3(t_2 \odot t_1)$, $C_5(t_1 \odot t_2) = C_5(t_2 \odot t_1)$; that is, the compositions C_1, C_3, C_5 are *commutative* under reversal of sequence. In fact, C_6 is also *commutative* if negative signs are ignored in the results. But C_2 and C_4 interchange results with the order of the first two terms reversed. See Table 3 below.

t_1	t_2	C_1	T_1	C_2	T_2	C_3	T_3	C_4	T_4	C_5	T_5	C_6	T_6
171	57	9747	9747	7980	7980	44232	5529	38497	38497	−14893	14893	34387	34387
140	176	64796	64796	70531	70531	24640	3080	30096	30096	38076	38076	−22116	22116
221	185	65525	65525	70981	70981	50632	6329	48865	48865	40885	40885	40885	40885
gcd		1	1	1	1	8	1	1	1	1	1	1	1
57	171	9747	9747	30096	30096	44232	5529	70531	70531	−14893	14893	34387	34387
176	140	64796	64796	38497	38497	24640	3080	7980	7980	38076	38076	22116	22116
185	221	65525	65525	48865	48865	50632	6329	70981	70981	40885	40885	40885	40885
gcd		1	1	1	1	8	1	1	1	1	1	1	1

Table 3. C_1, C_2, etc. are the composition formulas applied on $t_1 \odot t_2$; T_1, T_2, etc. as in Tables 1 & 2.

5. **Chain of compositions: Associativity check.**
Let $t_1 = (3, 4, 5), t_2 = (5, 12, 13), t_3 = (7, 24, 25), t_4 = (15, 8, 17)$. Let the chain of compositions $t_1 \odot t_2 \odot t_3 \odot t_4$ be made under all of $C_1, C_2, \ldots. C_6$. It is found that $C_1[\{C_1(t_1 \odot t_2)\} \odot \{C_1(t_3 \odot t_4)\}] = C_1[(15, 112, 113) \odot (105, 608, 617)] = (175, 15312, 15313)$. Also, $C_1[C_1\{C_1(t_1 \odot t_2)\} \odot t_3\}] \odot t_4] = C_1\{[C_1\{(15, 112, 113) \odot (7, 24, 25)\}] \odot (15, 8, 17)\} = C_1\{(105, 5512, 5513) \odot (15, 8, 17)\} = (175, 15312, 15313)$, which is the same as $C_1[\{C_1(t_1 \odot t_2)\} \odot \{C_1(t_3 \odot t_4)\}]$. This shows that a chain of compositions under C_1 is *associative*. Similarly, compositions under C_3, C_5 are also *associative*. Again, C_6 is also *associative* if negative signs are ignored in composition. Compositions under C_2 and C_4 only are *non-associative*.

6. **Negative values.** As discussed above and as will be seen in Table 4 below, keeping or ignoring negative signs produces quite a different picture in terms of the triples generated. So, strictly speaking, retaining the negative signs ought to be a more authentic process.

t_1	t_2	C_1	T_1	C_2	T_2	C_3	T_3	C_4	T_4	C_5	T_5	C_6	T_6
3	3	9	9	12	12	30	15	35	35	−7	7	25	1
4	4	40	40	35	35	16	8	12	12	24	24	0	0
5	5	41	41	37	37	34	17	37	37	25	25	25	1
gcd		1	1	1	1	2	1	1	1	1	1	25	1
−7	3	−21	21	72	72	40	5	195	195	−117	117	75	3
24	4	220	220	65	65	96	12	−28	28	44	44	100	4
25	5	221	221	97	97	104	13	197	197	125	125	125	5
gcd		1	1	1	1	8	1	1	1	1	1	25	1
7	3	21	21	72	8	110	55	195	195	−75	3	117	117
24	4	220	220	135	15	96	48	28	28	100	4	44	44
25	5	221	221	153	17	146	73	197	197	125	5	125	125
gcd		1	1	9	1	2	1	1	1	25	1	1	1

Table 4. C_1, C_2, etc. are the composition formulas applied on $t_1 \odot t_2$; T_1, T_2, etc. as in Tables 1 & 2.
(We have highlighted differences in values.)

7. **Return to the original triple.** From definition, $C_5[(a_1, b_1, c_1) \odot (a_2, b_2, c_2)] = (a_1a_2 - b_1b_2, b_1a_2 + a_1b_2, c_1c_2) = T$ (say). Now, $C_6[T \odot (a_1, b_1, c_1)] = c_1^2.\ (a_2, b_2, c_2)$, that is, the original triple (a_2, b_2, c_2) returns. Similarly, in $C_6[T \odot (a_2, b_2, c_2)] = c_2^2.\ (a_1, b_1, c_1)$, which indicates return of the original triple

(a_1, b_1, c_1). Example: $C_5[(7,4,5) \bullet (15,8,17)] = (-87, 416, 425)$. Then, $C_6[(-87, 416, 425) \bullet (15, 8, 17)] = (2023, 6936, 7225) \Rightarrow (7, 24, 25)$, dividing by $gcd = 17^2$; and $C_6[(-87, 416, 425) \bullet (7, 24, 25)] = (9375, 5000, 10625) \Rightarrow (15, 8, 17)$, dividing by $gcd = 25^2$.

8. **Non-Primitive PTs: the GCD.** It can be seen that in each of the composed triples, there is a product term like $a_1a_2, b_1a_2, b_1b_2, a_1b_2, c_1c_2$; the rest are either sums or differences of products, like $(b_1b_2 + c_1c_2) or (a_1a_2 - b_1b_2)$. In generating non-primitive triples, it is always the product term that plays the decisive role. *If $d > 1$, where d is a common divisor between the two factors of the product term, and certain other conditions are satisfied, then the composed triple will be a non-PPT.* In the discussion below, I will look into these *certain other conditions*.

   C_1: Take the triple $C_1[(a, b, c) \odot (x, y, z)] = (ax, bz + cy, by + cz)$. I assume that b and y are even. Let a common divisor, not necessarily the gcd, of (a, x) be $d > 1$. So ax contains the divisor d^2. Let $a = dma_0, x = dnx_0$. Let $dm > a_0, x_0 > dn$. Then, $b = \frac{1}{2}(d^2m^2 - a_0^2), c = \frac{1}{2}(d^2m^2 + a_0^2)$; and $y = \frac{1}{2}(x_0^2 - n^2d^2), z = \frac{1}{2}(x_0^2 + n^2d^2)$. So, $ax = d^2mna_0x_0$; $bz + cy = \frac{1}{2}d^2(m^2x_0^2 - n^2a_0^2)$; and, $by + cz = \frac{1}{2}d^2(m^2x_0^2 + n^2a_0^2)$. As a and x are both odd, so in the composed triple, the gcd $(ax, bz + cy, by + cz) = d^2$. *The necessary condition here is $dm > a_0, x_0 > dn$; if it is not satisfied, the* $gcd = 1$. Thus, $C_1[(15, 8, 17) \odot (255, 32, 257)] = (3825, 2600, 4625)$, a non-PPT with $gcd = 5^2$. Again, $C_1[(3, 4, 5) \odot (21, 220, 221)] = (63, 1984, 1985)$, a PPT; whereas $C_1[(3, 4, 5) \odot (21, 20, 29)] = (63, 216, 225)$, a non-PPT with $gcd = 3^2$.

   C_2: Now take $C_2[(a, b, c) \odot (x, y, z)] = (bx, az + cy, ay + cz)$. Let $b = dmb_0, x = dnx_0$. With b even and x odd, d, n, x_0 should be odd. So, $a = b_0^2 - (\frac{md}{2})^2, c = b_0^2 + (\frac{md}{2})^2$; and $y = \frac{1}{2}(d^2n^2 - x_0^2)$, $z = \frac{1}{2}(d^2n^2 + x_0^2)$. Thus, $(bx, az + cy, ay + cz) = [d^2mnb_0x_0, \frac{1}{4}d^2(4b_0^2n^2 - m^2x_0^2), \frac{1}{4}d^2(4b_0^2n^2 + m^2x_0^2]$. To make md even, as d is odd, m should be even; and that also makes the second and third terms integers. Note: Here the necessary condition is $b_0 > \frac{md}{2}, dn > x_0$, and m even. Thus, $C_2[(11, 60, 61) \odot (15, 8, 17)] = (900, 675, 1125)$, a non-PPT with $gcd = 225$.

   C_4: Similar to C_2 will be the case for $C_4[(a, b, c) \odot (x, y, z)] = (bz + cx, ay, bx + cz)$.

   C_3: In $C_3[(a, b, c) \odot (x, y, z)] = (az + cx, by, ax + cz)$ the terms are all even; so, there is always a common divisor 2. But, let $b = dmb_0, y = dny_0$. Here, if the condition $a = b_0^2 - (\frac{md}{2})^2$, $c = b_0^2 + (\frac{md}{2})^2$, and $x = (nd)^2 - (\frac{y_0}{2})^2, z = (nd)^2 + (\frac{y_0}{2})^2$, is satisfied, then $az + cx = \frac{1}{8}d^2(16n^2b_0^2 - m^2y_0^2), by = d^2mnb_0y_0, ax + cz = \frac{1}{8}d^2(16n^2b_0^2 + m^2y_0^2)$. Here the necessary conditions are: md and y_0 are even, and $b_0 > \frac{md}{2}, nd > \frac{y_0}{2}$. So, $gcd(az + cx, by, ax + cz) \geq 2$. Thus, $C_3[(7, 24, 25) \odot (35, 12, 37)] = (1134, 288, 1170)$, with $gcd = 18$; and, $C_3[(5, 12, 13) \odot (7, 24, 25)] = (216, 288, 360)$, which has a $gcd = 72$. But $C_3[(5, 12, 13) \odot (9, 40, 41)] = (322, 480, 578)$ has $gcd = 2$.

   C_5 & C_6: If for (a, b, c) and (x, y, z) the $gcd(c, z) = d > 1$, then a non-PPT with $gcd = d^2$ between the terms will be generated by either C_5 or C_6, never both. Obviously, C_5 and C_6 generate different triples with the same term cz. Thus, for $(5, 12, 13) \odot (33, 56, 65)$, C_5 generates $(-507, 676, 845)$, with $gcd = 13^2$ (if negative signs are ignored); and C_6 generates $(837, 116, 845)$, a PPT. But for $(5, 12, 13) \odot (63, 16, 65)$, C_5 generates $(123, 836, 845)$, a PPT; while C_6 gives $(507, 676, 845)$, with $gcd = 13^2$.

9. **Recurrence of triples.** The discussion on properties shows that in a process of composition there will be recurrence of triples (as exemplified by Tables 1 - 4). Recurrence of triples will be caused by:

(i) Reversal of the order of the first two terms of any one or both of the composing triples. For instance, (12, 35, 37) and (35, 12, 37) were both generated in the first row in Table 1, and both were used in subsequent compositions.

(ii) Reversal of the sequence of composition of the composing triples: that is, by *commutativity*, wherever it exists.

(iii) *Associativity*, wherever it exists: that is, generating the same triple through different sequences in a given chain of compositions.

[(ii) and (iii) can happen when all possible sequences of compositions are tried during the process.]

(iv) Return to the original triple.

(v) Most importantly, when non-PPTs are reduced to PPTs: that is, when the PPT thus generated may have been generated earlier or will be generated later in the composition process by composing different triples.

Infinitely many triples and infinitely many recurrences

Let us imagine a continuous composition process that starts with one triple (a PPT), which is composed with itself; and all non-primitive PTs that are generated are reduced to their corresponding PPTs. Then every PPT thus produced is composed with itself and with every other triple; and the process goes on indefinitely, and all possible sequences of composition are admitted. What is the outcome of such a process?

It has been seen that:

1. The formulas C_1 to C_6 constitute a system by which infinitely many Pythagorean triples can be generated by compositions, starting from just one triple.
2. Triples generated by the composition formulas are not all unique; occasionally the triples recur.

As the number of compositions increases, the number of recurrences is also likely to increase. But that will depend on the compositions chosen. The big question is: *Will there be a point of 'saturation' when triples are merely repeated and no new ones are generated?*

But we have seen that recurrences occur only when very specific conditions obtain. So, at every composition there will be as much possibility of a new triple generation as of the recurrence of an old one. Therefore, the point of '*saturation*' will never come to be, because *while infinitely many new triples will be generated, there will also be infinitely many recurrences, but this will be a never-ending process*.

What will happen is that:

1. *Over a finite range of compositions, the number of recurrences of triples will vary according to the compositions chosen.*
2. *Over an infinite range of compositions, while infinitely many new triples will be generated, there will also be infinitely many recurrences; but that being a never-ending process, no point of 'saturation' when new triples cease to be generated will ever be reached.*

Appendix

The Compositions on an EXCEL sheet

This is how the Tables have been created on EXCEL. The following table is a representation of one set of compositions ($C_1 - C_6$) in which the cells are marked as A1, A2, A3, B1, B2, B3, etc. The EXCEL formulas in each cell are shown below the table.

A1	B1	C1	D1	E1	F1	G1	H1	I1	J1	K1	L1	M1	N1
A2	B2	C2	D2	E2	F2	G2	H2	I2	J2	K2	L2	M2	N2
A3	B3	C3	D3	E3	F3	G3	H3	I3	J3	K3	L3	M3	N3
gcd		C4	D4	E4	F4	G4	H4	I4	J4	K4	L4	M4	N4

Triple (a_1, b_1, c_1) is laid out in the cells: A1: a_1; A2: b_1; A3: c_1

Triple (a_2, b_2, c_2) is laid out in the cells: B1: a_2; B2: b_2; B3: c_2

C_1: $[a_1a_2, (b_1c_2 + c_1b_2), (c_1c_2 + b_1b_2)]$
C1 = A1*B1; C2 = A2*B3+B2*A3; C3 = A2*B2+A3*B3; C4 = GCD(ABS(C1),ABS(C2),ABS(C3))
D1 = ABS(C1)/C4; D2 = ABS(C2)/C4; D3 = ABS(C3)/C4; D4 = GCD(D1,D2,D3)

C_2: $[b_1a_2, (a_1c_2 + c_1b_2), (c_1c_2 + a_1b_2)]$
E1 = A2*B1; E2 = A1*B3+A3*B2; E3 = A1*B2+A3*B3; E4 = GCD(ABS(E1),ABS(E2),ABS(E3))
F1 = ABS(E1)/E4; F2 = ABS(E2)/E4; F3 = ABS(E3)/E4; F4 = GCD(F1,F2,F3)

C_3: $[(a_1c_2 + c_1a_2), b_1b_2, (c_1c_2 + a_1a_2)]$
G1 = A1*B3+A3*B1; G2 = A2*B2; G3 = A1*B1+A3*B3; G4 = GCD(ABS(G1),ABS(G2),ABS(G3))
H1 = ABS(G1)/G4; H2 = ABS(G2)/G4; H3 = ABS(G3)/G4; H4 = GCD(H1,H2,H3)

C_4: $[(b_1c_2 + c_1a_2), a_1b_2, (c_1c_2 + b_1a_2)]$
I1 = A2*B3+A3*B1; I2 = A1*B2; I3 = A2*B1+A3*B3; I4 = GCD(ABS(I1),ABS(I2),ABS(I3))
J1 = ABS(I1)/I4; J2 = ABS(I2)/I4; J3 = ABS(I3)/I4; J4 = GCD(J1,J2,J3)

C_5: $[(a_1a_2 - b_1b_2), (a_1b_2 + b_1a_2), c_1c_2]$
K1 = A1*B1−A2*B2; K2 = A1*B2+A2*B1; K3 = A3*B3; K4 = GCD(ABS(K1),ABS(K2),ABS(K3))
L1 = ABS(K1)/K4; L2 = ABS(K2)/K4; L3 = ABS(K3)/K4; L4 = GCD(L1,L2,L3)

C_6: $[(a_1a_2+ b_1b_2), (b_1a_2- a_1b_2), c_1c_2]$
M1 = A1*B1+A2*B2; M2 = A2*B1−A1*B2; M3 = A3*B3; M4 = GCD(ABS(M1),ABS(M2),ABS(M3))
N1 = ABS(M1)/M4; N2 = ABS(M2)/M4; N3 = ABS(M3)/M4; N4 = GCD(N1,N2,N3)

BODHIDEEP JOARDAR (born 2005) is a student of South Point High School, Calcutta who is a voracious reader of all kinds of mathematical literature. He is interested in number theory, Euclidean geometry, higher algebra, foundations of calculus and infinite series. He feels inspired by the history of mathematics and by the lives of mathematicians. His other interests are in physics, astronomy, painting and the German language. He may be contacted at ch_kakoli@yahoo.com.