Problem Corner

Problems for the SENIOR SCHOOL

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Problem VIII-1-S.1

An altitude AH of triangle ABC bisects a median BM. Prove that the medians of the triangle ABM are side-lengths of a right-angled triangle.

Problem VIII-1-S.2

There exists a block of 1000 consecutive positive integers containing no prime numbers, namely, $1001!+2, 1001!+3, \ldots, 1001!+1001$. Does there exist a block of 1000 consecutive positive integers containing exactly 5 prime numbers?

Problem VIII-1-S.3

Initially, the number 1 and two positive numbers x and y are written on a blackboard. In each step, we can choose two numbers on the blackboard, not necessarily different, and write their sum or their difference on the blackboard. We can also choose a non-zero number on the blackboard and write its reciprocal on the blackboard. Is it possible to write on the blackboard, in a finite number of moves, the numbers x^2 and xy?

Problem VIII-1-S.4

For which positive integers n can one find distinct positive integers $a_1, a_2, \ldots, a_n$ such that the number

$$N = \frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is also an integer?

Problem VIII-1-S.5

In triangle ABC, $\measuredangle A = 2\measuredangle B = 4\measuredangle C$. Their bisectors meet the opposite sides at D, E and F respectively. Prove that $DE = DF$.

Keywords: Altitude, median, prime number

Solutions of Problems in Issue-VII-3 (November 2018)

Solution to problem VII-3-S.1

Let $f(x) = x^2 + bx + c$ where b is a negative integer and c is a real number. Suppose the sum of the roots of $f(f(x))$ is a prime number. Prove that $f(f(x))$ has no real root in the interval $(0, 1)$.

Observe that

$$f(f(x)) = x^4 + 2bx^3 + (b^2 + b + 2c)x^2 + (2bc + b^2)x + (c^2 + bc + c).$$

Since the sum of roots $-2b$ is a prime number, $b = -1$. Hence

$$f(f(x)) = x^4 - 2x^3 + 2cx^2 + (1 - 2c)x + c^2 = (x(x-1) + c)^2 + x(1-x)$$

and this is positive for $x \in (0, 1)$.

Solution to problem VII-3-S.2

Let k be a given positive integer. Determine all real x, y, z such that $xyz \neq 0$ and

$$x^k + y^{k+1} = z^{k+2}, \quad x^{k+1} + y^{k+2} = z^{k+3}, \quad x^{k+2} + y^{k+3} = z^{k+4}.$$

Observe that

$$\left(x^k + y^{k+1}\right)\left(x^{k+2} + y^{k+3}\right) = z^{k+2} \cdot z^{k+4} = \left(z^{k+3}\right)^2 = (x^{k+1} + y^{k+2})^2,$$

which implies

$$x^k y^{k+1}\left(x^2 - 2xy + y^2\right) = 0,$$

i.e., $x = y$ (since $xyz \neq 0$). Let $x = y = t$. Then $t \neq 0$ and

$$z = \frac{z^{k+3}}{z^{k+2}} = \frac{t^{k+1}(1+t)}{t^k(1+t)} = t.$$

Thus $1 + t = t^2$ and $t = \dfrac{1-\sqrt{5}}{2}, \dfrac{1+\sqrt{5}}{2}$. Hence

$$(x, y, z) = \left(\frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right), \quad \left(\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right).$$

There are two solutions.

Solution to problem VII-3-S.3

A quadratic polynomial $f(x) = ax^2 + bx + c$ has no real roots. It is given that b is a rational number, and exactly one of c and $f(c)$ is a rational number. Is it possible for the discriminant of $f(x)$ to be a rational number? [Russian Mathematical Olympiad]

Suppose that c is a rational number. Then, by hypothesis, $f(c) = c(ac + b + 1)$ is irrational. Since b and c are rational, a must be irrational. Therefore the discriminant $D = b^2 - 4ac$ is irrational.

Suppose that $f(c)$ is rational but c is irrational. Note $f(c) \neq 0$, since f does not have any real root. Then $(ac + b + 1) \neq 0$ and is irrational. But b is rational. Therefore ac is irrational and hence $D = b^2 - 4ac$ is irrational.

It follows that D is irrational.

Solution to problem VII-3-S.4

The sequence $\{a_n\}_{n\geq 0}$ is defined as follows:

$$a_0 = 1, \quad a_1 = 3, \quad a_{n+1} = a_n + a_{n-1} \text{ for all } n \geq 1.$$

Find all integers $n \geq 1$ for which $na_{n+1} + a_n$ and $na_n + a_{n-1}$ share a common factor greater than 1.

Let $d > 1$ be a common factor of $na_{n+1} + a_n$ and $na_n + a_{n-1}$ for some $n \geq 1$. Then we see that d divides

$$(na_{n+1} + a_n) - (na_n + a_{n-1}) = na_{n-1} + a_{n-2}.$$

Continuing this way we see that d divides $na_1 + a_0$ and $na_2 + a_1$. Thus d divides both

$$n(a_2 - a_1) + (a_1 - a_0) = n + 2$$

and

$$3(na_2 + a_1) - 4(na_1 + a_0) = 3(4n + 3) - 4(3n + 1) = 5.$$

Since $d > 1$ and 5 is prime, $d = 5$ and $n + 2$ is a multiple of 5. Thus for any n of the form $5k + 3$, where k is a non-negative integer, $na_{n+1} + a_n$ and $na_n + a_{n-1}$ share a common factor greater than 1.

Conversely, if n is of the form $5k + 3$ for some non-negative integer k, then $n + 2$ is divisible by 5 and retracing the steps we see that $na_{n+1} + a_n$ and $na_n + a_{n-1}$ are both divisible by 5.

Solution to problem VII-3-S.5

Consider the sequence $\{10^n\}_{n\geq 1}$. Prove that the sum of no two terms of the sequence is a perfect square.

Let $a_n = 10^n$ for $n \geq 1$. Then $a_n \equiv 1 \pmod 3$ for all n, hence

$$a_j + a_k \equiv 2 \pmod 3.$$

Since the square of any integer is either $0 \pmod 3$ or $1 \pmod 3$, a number of the form $2 \pmod 3$ is never a perfect square. Therefore $a_j + a_k$ is never a perfect square for any $1 \leq j \leq k$.