Problem Corner

Problems for the MIDDLE SCHOOL

Problems in Pure Geometry

A RAMACHANDRAN

Pure geometry or Euclidean geometry is a body of theorems and corollaries logically derived from certain axioms and postulates as presented in Euclid's *Elements*. Later geometers, both Greek and others, have added to this. Occasionally some algebra is brought in but not trigonometry. Abraham Lincoln is said to have read the *Elements* just for the reasoning.

Problems

Problem VIII–1–M.1

A fallacy is an argument that sounds logical but is actually erroneous. So the conclusion drawn is false. Often the error is not easy to spot – these could be erroneous assumptions, unwarranted generalisations or applying logic out of context. Fallacies are fun to go through and can help us gain greater alertness in reasoning. Here is a fallacy in geometry – a 'proof' that "all triangles are equilateral." The question is to spot the error.

In an arbitrary triangle ABC (Figure 1) the bisector of $\measuredangle A$ meets the perpendicular bisector of side BC at point O. Join OB and OC. OB $=$ OC as any point on the perpendicular bisector of a line segment is equidistant from the two ends of the line segment.

Now drop perpendiculars OY and OZ from O to AC and AB respectively. OY $=$ OZ as any point on an angle bisector is equidistant from the arms forming the angle.

Now consider ΔBOZ and ΔCOY. We have OB $=$ OC, OZ $=$ OY, and $\measuredangle BZO = \measuredangle CYO$ (right angles).

So ΔBOZ $\cong$ ΔCOY (RHS congruency), and therefore BZ $=$ CY. But we have AZ $=$ AY and so AB $=$ AC.

Similarly we could show that AC = BC and thereby claim that ΔABC is equilateral.

Keywords: Pure geometry, fallacy, proof

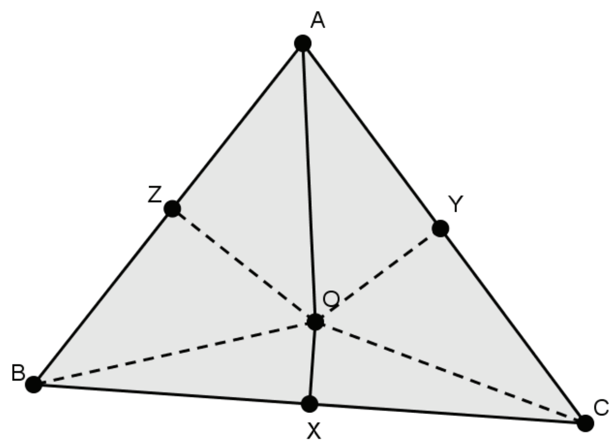

Figure 1

Problem VIII–1–M.2

Figure 2 shows two triangles ΔABC and ΔPQR. $AB = PQ$, $AC = PR$, and $\measuredangle BAC$ and $\measuredangle QPR$ are supplementary.

Give a geometric proof that the triangles are equal in area.

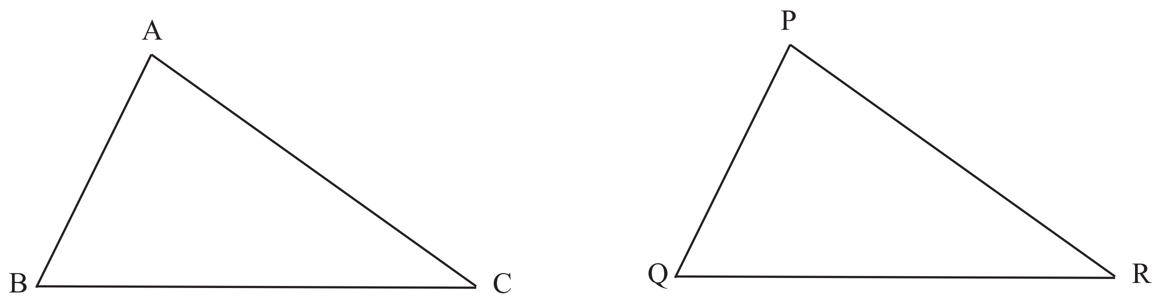

Figure 2

Problem VIII–1–M.3

Figure 3 shows a rectangle composed of three squares with some additional lines drawn. Give a geometric proof that $\measuredangle EAD + \measuredangle EBD = \measuredangle ECD (= 45^\circ)$.

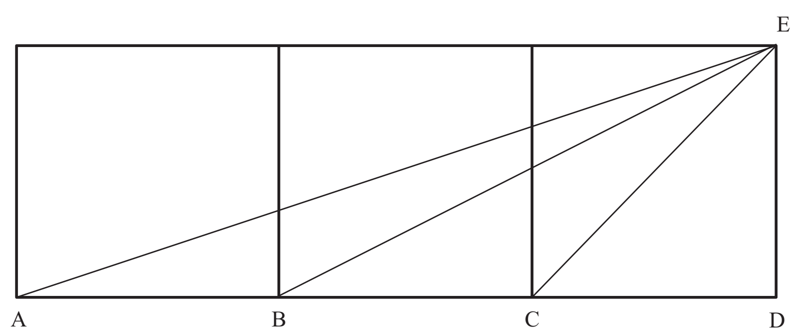

Figure 3

Problem VIII–1–M.4

Figure 4 shows three semicircles on a shared base and on the same side of it. The sum of the diameters (or radii) of the two smaller semicircles equals the diameter (or radius) of the largest semicircle ($BA + AC = BC$). We can consider the radius of the largest semicircle to be unity and that of one of the smaller to be r, while the radius of the other is $1 - r$. AH is drawn perpendicular to BC, with H on the largest semicircle. HB intersects one of the smaller semicircles at D, while HC intersects the other at E. DE intersects AH at O.

Prove the following:

(a) The area of the circle with diameter AH equals the area of the region enclosed by the three semicircles (shaded blue).

(b) AH and DE are equal in length and bisect each other.

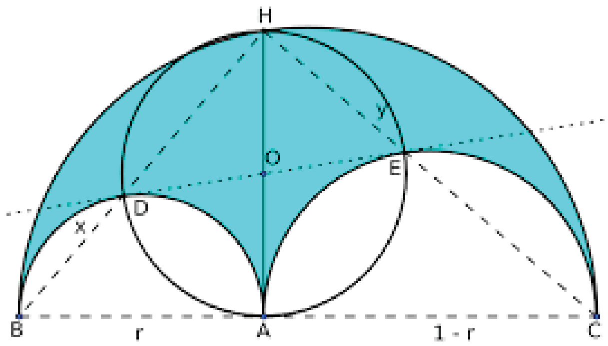

Figure 4

Problem VIII–1–M.5

The angles at the 5 corners of a pentagram (5-pointed star) total to 180°. This is easy to prove for a symmetrical figure. The central part is a regular pentagon. Knowing its interior angle to be 108° one can see that each base angle of the isosceles triangles is 72° and hence calculate the angle at each apex to be 36°. So the 5 apex angles add to 180°. The question here is to prove the same for any (asymmetrical) pentagram.

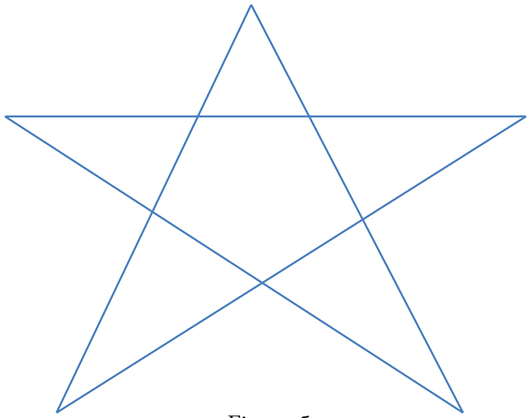

Figure 5

Solutions

Problem VIII-1-M.1

The fallacy lies in the assumption that the bisector of $\measuredangle A$ and the perpendicular bisector of BC meet inside the triangle. If $AB \neq AC$, they meet outside the triangle. This can be seen if we note that the bisector of $\measuredangle A$ divides BC in the ratio AB : AC. Hence if $AB < AC$ say, then bisector of $\measuredangle A$ will meet BC at a point closer to B than to C. When extended it will meet the perpendicular bisector of BC at point O outside the triangle. Do make an actual construction and observe the following (Figure 1.1):

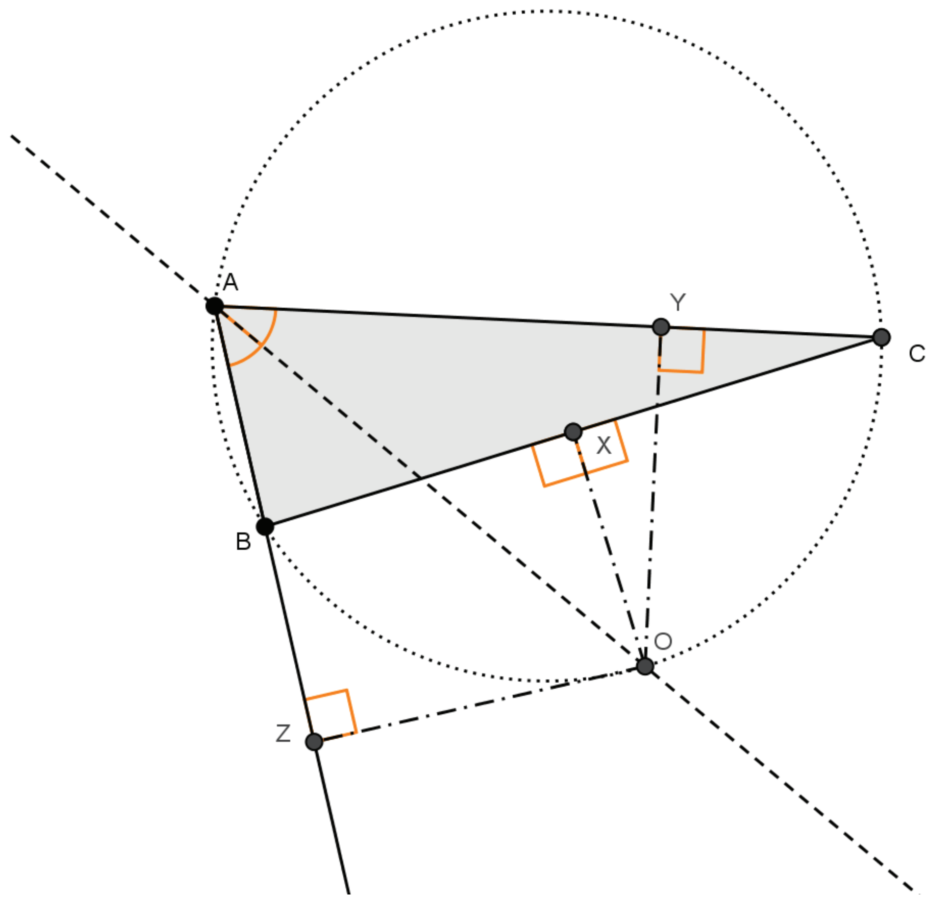

Figure 1.1

Point O lies on the circumcircle of ΔABC.

Point Y lies on AC while point Z lies on AB extended.

Congruent triangles COY and BOZ do get formed but we have $AC - CY = AB + BZ$.

If $AB = AC$, the two lines merge into a single line of symmetry.

Figure 1.1 does not represent a valid geometrical situation and therein lies the fallacy.

Additional comment from the editor. The reader may wish to explore with this configuration further, as there is a subtle issue involved. What we need to show is that (as in Figure 1.1), it can never happen that both Y and Z lie in the interiors of AC and AB respectively, or that both lie on the extensions of these sides respectively. It will always happen that one of the points lies in the interior of its respective side and the other point lies on the extension of the side. We need to show that this will *always* be the case. (If a situation occurred when both Y and Z lie on the extensions of their respective sides, the original reasoning would continue to work and the same fallacy would result.) We invite the reader to explore further.

Problem VIII-1-M.2

Figure 2 shows two triangles ΔABC and ΔPQR. $AB = PQ$, $AC = PR$, and $\measuredangle BAC$ and $\measuredangle QPR$ are supplementary.

Give a geometric proof that the triangles are equal in area.

Solution. To prove that ΔABC and ΔPQR are of equal area we carry out the following construction. Extend CA to point D such that $AC = AD$ (Figure 2.1). Join BD.

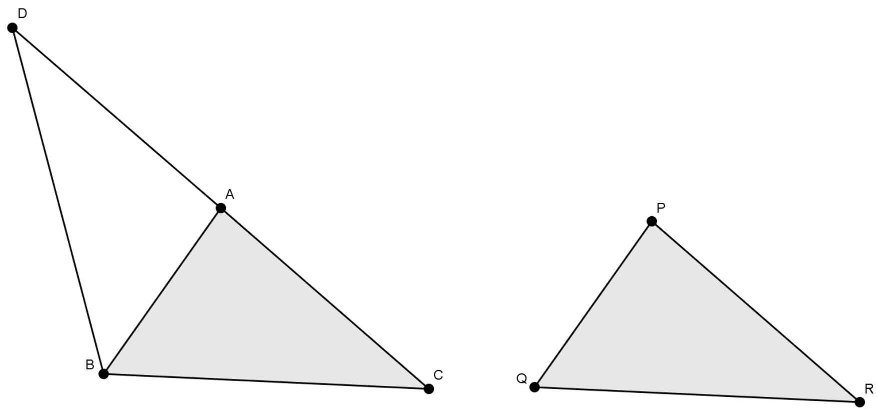

Figure 2.1

Considering ΔABD and ΔPQR, we have

$$AB = PQ,\ AD = AC = PR \text{ and } \measuredangle DAB = 180^\circ - \measuredangle BAC = \measuredangle QPR.$$

Hence the triangles are congruent. Since BA is a median of ΔBCD, ΔABC, ΔABD and therefore ΔPQR are of equal area.

Problem VIII-1-M.3

Figure 3 shows a rectangle composed of three squares with some additional lines drawn. Give a geometric proof that $\measuredangle EAD + \measuredangle EBD = \measuredangle ECD (= 45^\circ)$.

Solution. We take the squares in Figure 3 to have sides of unit length. In ΔECA the side lengths, in increasing order, are $\sqrt{2}$, 2, $\sqrt{10}$, while the side lengths of ΔBCE are 1, $\sqrt{2}$, $\sqrt{5}$. That is, $CE/BC = AC/EC = AE/BE$. As the corresponding lengths are in proportion, the triangles are similar and the angles opposite corresponding sides are equal. Hence $\measuredangle EAC = \measuredangle BEC$. Therefore $\measuredangle EAD + \measuredangle EBD = \measuredangle BEC + \measuredangle EBD = \measuredangle ECD = 45^\circ$.

Problem VIII-1-M.4

Figure 4 shows three semicircles on a shared base and on the same side of it. The sum of the diameters (or radii) of the two smaller semicircles equals the diameter (or radius) of the largest semicircle ($BA + AC = BC$). We can consider the radius of the largest semicircle to be unity and that of one of the smaller to be r, while the radius of the other is $1 - r$. AH is drawn perpendicular to BC, with H on the largest semicircle. HB intersects one of the smaller semicircles at D, while HC intersects the other at E. DE intersects AH at O.

Prove the following:

(a) The area of the circle with diameter AH equals the area of the region enclosed by the three semicircles (shaded blue).

(b) AH and DE are equal in length and bisect each other.

Solution.

(a) It is a well-known result that the length AH is the geometric mean of the lengths BA and AC. You may be aware that the geometric mean of two positive quantities a and b is $\sqrt{ab}$. As mentioned earlier, we take the radius of the largest semicircle to be unity and the radius of the semicircle BDA to be r. Then length BA $= 2r$, while length AC $= 2(1-r)$. Then AH $= \sqrt{[2r \cdot 2(1-r)]} = 2\sqrt{r(1-r)}$. The area of the circle with AH as diameter would then be $\pi r(1-r)$.

Area of region enclosed by the three semicircles is area of large semicircle diminished by sum of areas of the other two, which is $\frac{\pi}{2} - \frac{\pi}{2}(r^2 + (1-r)^2)$ which simplifies to $\pi r(1-r)$.

(b) $\measuredangle$BHC, $\measuredangle$BDA and $\measuredangle$AEC are all right angles being angles in semicircles, making quadrilateral HDAE a rectangle. Hence HA and DE are equal and bisect each other.

Figure 4 was studied by the ancient Greeks. Archimedes named it "Arbelos," which is the Greek word for the 'shoemaker's knife,' which the figure resembles.

Problem VIII-1-M.5

The angles at the 5 corners of a pentagram (5-pointed star) total to 180°. This is easy to prove for a symmetrical figure. The central part is a regular pentagon. Knowing its interior angle one can calculate the angle at the apex of any of the triangles in the figure. The question here is to prove the same for any (asymmetrical) pentagram.

Solution. One way to prove the above statement is to note that the 10 angles at the bases of the 5 triangles form two sets of 5 exterior angles of the central pentagon. So they add up to $2 \times 360° = 720°$. The sum of all the angles of the 5 triangles would total $5 \times 180° = 900°$. So the apex angles should add up to 180°. An alternative proof is given below as a self-explanatory figure.

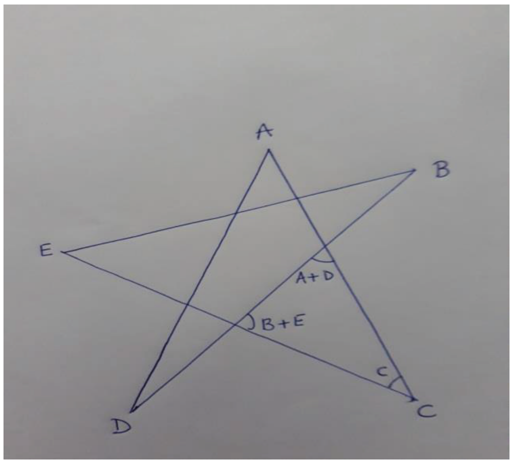

Figure 6

Acknowledgements. Figure 4 is taken from Wikipedia.