At Right Angles
A Resource for School Mathematics

INTRODUCTION TO ALGEBRA - IV

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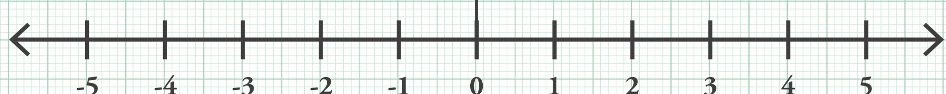

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INDICES AND IDENTITIES

This is the fourth article in the series, *Introduction to Algebra*. The first one looked at algebra as a 'pattern language' and focused on noticing patterns and expressing patterns using terms and expressions. The second one looked at algebra as a 'design language' and focused on expressing designs using algebraic language. The third one explored solving simple equations using the 'balance approach' and the 'machine approach.' However, the topic of equations is vast and gets steadily more complex as the student progresses through higher algebra. Similarly, the topic of solving and simplifying expressions of higher degree steadily grows in complexity.

The precursor to simplification, factorisation and expansion of expressions of higher degree is a study of the laws of indices and basic identities. The index laws come into play while using large numbers as well, which are often written in scientific notation and involve powers. When these numbers are multiplied, divided or raised to a power, the laws of indices are applied.

In this article, we study indices (only positive whole number indices) and basic identities. The article does not go into the application of these concepts in problem situations.

Note: While getting students to learn concepts, it is crucial to cover only one aspect at a time, build it up gradually and not bring in several concepts or variations all at once.

ACTIVITY 1

Objective: Expressing higher powers of 10 in exponential form
Purpose of writing large numbers in a compact form.

Prerequisites:

- Familiarity with large numbers extending to millions and crores.
- Prime factorisation.

Let students bring some data collected from newspapers or books (atlas, geography book) which makes use of large numbers (rounded to a suitable multiple of a power of 10).

Check if they are comfortable reading and writing large numbers.

Select some large figures to write on the board and discuss the difficulty of writing and reading such numbers.

Example: The Sun is at a distance of **150,000,000** km from Earth.

Proxima Centauri, the closest star to our Sun, is **40,208,000,000,000** km away.

Show them how to represent these numbers using powers.

1,00,000 can be written as
$10 \times 10 \times 10 \times 10 \times 10$. This can be written as 10^5.

It is highly important to read it out aloud as '10 raised to the power of 5'.

Students have all along been used to seeing a sign between two numbers. This is their first encounter with two numbers where there is no explicit sign. It is going to take some time for the students to internalise this representation and interpret it correctly.

20,00,00,000 can be written as
$2 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$.

This can be written as 2×10^8. Read it out as '2 times 10 raised to the power of 8'. Point out that it is not the same as $(2 \times 10)^8$.

What would this expand into?
$20 \times 20 \times 20 \times 20 \times 20 \times 20 \times 20 \times 20$.

4,000,000,000 can be written as
$4 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$.

This can be written as 4×10^9. Read it out as '4 times 10 raised to the power of 9'.

Note: Formal words like *base* and *index* can be brought in a little later.

ACTIVITY 2

Objective: To contrast $a \times 3$ and a^3

Let students study the following pattern and describe it using pattern language.

- $5 + 5 + 5 = 15 = 3 \times 5$
- $2 + 2 + 2 = 6 = 3 \times 2$
- $7 + 7 + 7 = 21 = 3 \times 7$
- $a + a + a = 3a = 3 \times a$

Now let them study this pattern.

- $5 \times 5 \times 5 = 125$
- $2 \times 2 \times 2 = 8$
- $7 \times 7 \times 7 = 343$
- $a \times a \times a =$??? (Are the students able to say what will come here?)

Let students state the difference that they notice between these two patterns. The first one is an additive relationship where a quantity is repeatedly added to itself, whereas the second one is a multiplicative relationship where a quantity is repeatedly multiplied by itself.

The teacher can then show the standard form of writing and reading it.

- $5 \times 5 \times 5 = 125 = 5^3$ is read as '5 raised to the power of 3'.
- $2 \times 2 \times 2 = 8 = 2^3$ is read as '2 raised to the power of 3'.

- $7 \times 7 \times 7 = 343 = 7^3$ is read as '7 raised to the power of 3'.
- $a \times a \times a = a^3$ is read as 'a raised to the power of 3'.

Caution: Evaluating 4^3 as 4×3 is a very common mistake that occurs in the initial stage. This is because the students have not fully internalised the meaning of a^3.

Let there be plenty of oral exercises which reinforce the meaning of 'to the power of'.

Thirty second exercises which require mental calculations can be used to calculate powers of small numbers.

Teacher says '3 to the power of 4'. Students have to give the value within thirty seconds.

Teacher says '64'. Students have to respond using the exponential form, '2 to the power of 6' or '4 to the power of 3' or '8 to the power of 2.'

Visualising indices

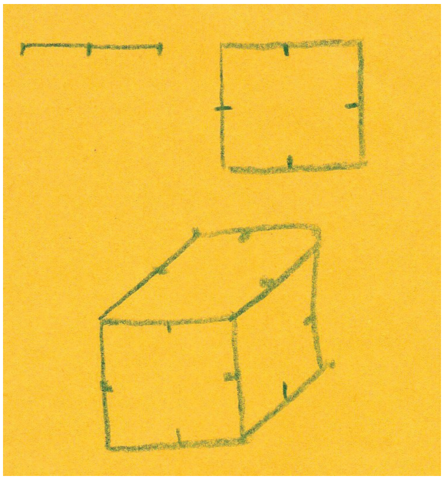

Figure 1

2^1 can be thought of as a line of length 2.

2^2 can be thought of as a square of side 2.

2^3 can be thought of as a cube of edge 2.

How does one visualise 2^4?

Here is one way!

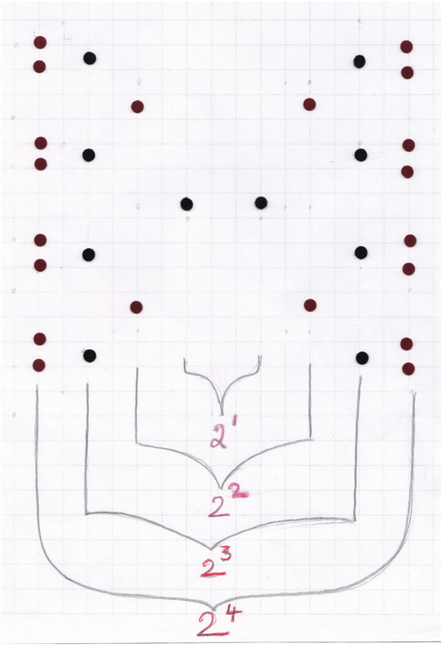

Figure 2

ACTIVITY 3

Objective: Exploration of numbers using indices

Prior knowledge: Prime factorisation

Ask students to find a number which can be expressed using indices in two ways.

Example: $16 = 2^4 = 4^2$.

How many such numbers can be found between 1 and 100?

Ask students to find a number which can be expressed using indices in three ways.

How many such numbers can be found between 1 and 100?

Which number between 1 and 200 can be so expressed in the largest number of ways?

Can the students pose more such challenges for themselves?

Is there any pattern to be found in such an exploration?

Game 1

Objective: Practice of index notation

Materials: Set of 16 cards

Prepare a set of 16 cards (8 matching pairs) where each pair consists of a number in index form and another its value.

(**Example:** 2^4, 16, $4 \times 4 \times 4$, 2^6, $3 + 3 + 3$, 3^2, 4^2)

The set of cards can now be placed face down and played as a memory game.

Note: Take care not to use power 0 until it has been properly introduced.

ACTIVITY 4

Objective: To study index laws when the base is common, $a^m \times a^n = a^{(m+n)}$

Let students study each of the following examples and discover the law.

Ex. 1. What does $5^3 \times 5^2$ become?

5 repeated as a factor 3 times × 5 repeated as a factor 2 times = 5 repeated as a factor (3 + 2) times.

$(5 \times 5 \times 5) \times (5 \times 5) = 5 \times 5 \times 5 \times 5 \times 5$

$5^3 \times 5^2 = 5^{(3+2)} = 5^5$

Ex. 2. What does $2^4 \times 2^5$ become?

Ex. 3. What does $3^4 \times 3^3$ become?

Let the students note down the result as given below, notice and express the pattern in a generalized manner.

- $5^3 \times 5^2 = 5^{(3+2)}$
- $2^4 \times 2^5 = 2^{(4+5)}$
- $3^4 \times 3^3 = 3^{(4+3)}$
- $a^m \times a^n = a^{(m+n)}$

Caution: Students often make mistakes in applying this law.

They need to be very clear that $a^m \times a^n$ is **not** equal to $a^m + a^n$.

It is important to state what $5^3 \times 5^2$ represents, as has been done earlier (5 repeated as a factor 3 times multiplied by 5 repeated as a factor 2 times = 5 repeated as a factor (3 + 2) or 5 times).

This can be done in the context of several examples, to avoid the above kind of mistake.

The teacher needs to help students to focus on the factor being repeated and the number of times it is being repeated.

Another point which needs to be made clear is that $a^m + a^n$ is not equal to $a^{(m+n)}$.

- What approaches can help to minimise these errors?

- Would visuals help? Would drill exercises help? Would true or false questions help?

I have found it useful to make a collection of such common errors and ask groups of students to discuss these errors among themselves, one type of error per group, and then share what was discussed with the rest of the class, giving reasons for its incorrectness.

Students can also verify such results by computations with small values.

Note: Students can now be introduced to the words *base* and *index*.

The **base** is the quantity that is repeated.

The **index** or **power** shows the number of times that quantity is repeatedly multiplied by itself. If it becomes confusing for a student to use two different words for the same thing, the teacher can stick to one of them.

At whichever point one introduces *coefficient*, care should be taken to see that the students do not confuse *coefficient* with *index*.

ACTIVITY 5

Objective: To study index laws when the base is common: $a^m \div a^n = a^{(m-n)}$

Let students study each of the following examples and discover the law.

Ex. 1. What does $4^5 \div 4^2$ become?

4 repeated as a factor 5 times ÷ 4 repeated as a factor 2 times is the same as 4 repeated as a factor (5 – 2) = 3 times.

$$\frac{(4 \times 4 \times 4 \times 4 \times 4)}{(4 \times 4)} = 4 \times 4 \times 4$$

$4^5 \div 4^2 = 4^{(5-2)} = 4^3$

Ex. 2. What does $7^8 \div 7^3$ become?

Ex. 3. What does $3^4 \div 3^3$ become?

Let the students note down the result as given below, notice and express the pattern in a generalized manner.

- $4^5 \div 4^2 = 4^{(5-2)} = 4^3$
- $7^8 \div 7^3 = 7^{(8-3)} = 7^5$
- $3^4 \div 3^3 = 3^{(4-3)} = 3^1$
- $a^m \div a^n = a^{(m-n)}$

Some students may wonder what happens if there is a higher or equal power in the denominator. The teacher can tell them that this would be taken up soon.

Caution: Again, watch out for common mistakes and make sure that students understand this.

$a^m \div a^n$ is **not** equal to $a^m - a^n$ and $a^m - a^n$ is **not** equal to $a^{(m-n)}$

Occasionally, students make errors like $3^4 \times 2^3 = 6^7$. It is important to go back to basics in such situations and explain the misconception.

Give examples as well as non-examples to demonstrate laws of indices.

For instance, it is not possible to apply the laws of indices to $3^4 \times 2^3$.

ACTIVITY 6

Objective: To study index laws when the index is common, $a^m \times b^m = (a \times b)^m$

Let students study each of the following examples and discover the law by themselves.

Ex. 1. What is $3^3 \times 5^3$?

- What is $2^4 \times 3^4$?
- What is $6^2 \times 4^2$?

Let the students write them in an expanded form.

- $3^3 \times 5^3 = 3 \times 3 \times 3 \times 5 \times 5 \times 5$
- $2^4 \times 3^4 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 3$
- $6^2 \times 4^2 = 6 \times 6 \times 4 \times 4$

Using the associative and commutative laws, the first one can be rearranged as $3 \times 5 \times 3 \times 5 \times 3 \times 5$.

This can now be written as $15 \times 15 \times 15$ which is 15^3.

Let the students work out the other two in a similar manner.

- $3^3 \times 5^3 = 15^3$
- $2^4 \times 3^4 = 6^4$
- $6^2 \times 4^2 = 24^2$

Do the students see the pattern?

$$a^m \times b^m = (ab)^m.$$

ACTIVITY 7

Objective: To show $\dfrac{a^m}{b^m} = \left(\dfrac{a}{b}\right)^m$

Let students study each of the following examples and discover the law.

Ex. 1. What is $5^8 \div 2^8$?

- What is $3^7 \div 5^7$?
- What is $6^9 \div 2^9$?

Let the students write the expressions in an expanded form.

$$\frac{5^8}{2^8} = \frac{(5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5)}{(2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2)}$$

This can be written as

$$\frac{5}{2} \times \frac{5}{2} \times \frac{5}{2} \times \frac{5}{2} \times \frac{5}{2} \times \frac{5}{2} \times \frac{5}{2} \times \frac{5}{2} = \left(\frac{5}{2}\right)^8.$$

Similarly $\dfrac{3^7}{5^7}$ becomes $\left(\dfrac{3}{5}\right)^7$.

$\dfrac{6^9}{2^9}$ becomes $\left(\dfrac{6}{2}\right)^9$

We see that $\dfrac{a^m}{b^m} = \left(\dfrac{a}{b}\right)^m$.

ACTIVITY 8

Objective: To show $a^0 = 1$

Explanation 1:

This makes use of students' knowledge of fractions and the law $\frac{a^m}{a^n} = a^{(m-n)}$.

Ask students to work out the following problems using fractions.

- What is $\frac{3^4}{3^4}$?
- $\frac{(3 \times 3 \times 3 \times 3)}{(3 \times 3 \times 3 \times 3)}$
- What is $\frac{2^5}{2^5}$?
- What is $\frac{7^2}{7^2}$?

In each case, the students will notice that the answer is 1.

Now let them use the law $\frac{a^m}{a^n} = a^{(m-n)}$

Using that each of the above examples results in

- $\frac{3^4}{3^4} = 3^{(4-4)} = 3^0$
- $\frac{2^5}{2^5} = 2^{(5-5)} = 2^0$
- $\frac{7^2}{7^2} = 7^{(2-2)} = 7^0$

Using fractions, it has already been established that they are equal to 1.

Hence $3^0 = 1$

$2^0 = 1$

$7^0 = 1$

Generalising, we get $a^0 = 1$.

Explanation 2:

Let students write the values of each of these and notice the pattern.

- $2^5 = 32$
- $2^4 = 16$
- $2^3 = 8$
- $2^2 = 4$
- $2^1 = 2$
- $2^0 = ?$

Each successive answer is $\frac{1}{2}$ of the earlier answer.

- $\frac{1}{2}$ of 32 is 16.
- $\frac{1}{2}$ of 16 is 8.
- $\frac{1}{2}$ of 8 is 4.
- $\frac{1}{2}$ of 4 is 2.

Following the pattern, the next number needs to be $\frac{1}{2}$ of 2, which is 1.

Hence 2^0 is 1.

At a later point, the same approach can be used to handle negative indices, e.g., $2^{(-1)}$ or $3^{(-2)}$.

ACTIVITY 9

Objective: To extend the index laws to varied problems

The students' understanding and practice can be further enhanced by giving them problems of the following type to simplify.

- $a^l \times a^m \times a^n$
- $\frac{(a^c \times a^d)}{a^e}$
- $\frac{a^x}{(a^y \times a^z)}$

Varied problems involving comparison, sequencing and simplification can be given.

ACTIVITY 10

Objective: To extend the index laws for factors with coefficients.

Students need to be shown the difference between coefficients in this example.

Is $4x^2$ same as $(4x)^2$?

Is $-16x^2$ the same as $(-4x)^2$?

Proper usage of brackets and correct interpretation of the quantity being squared needs to be focused on.

ACTIVITY 11

Objective: To show the identity $(a + b)^2$

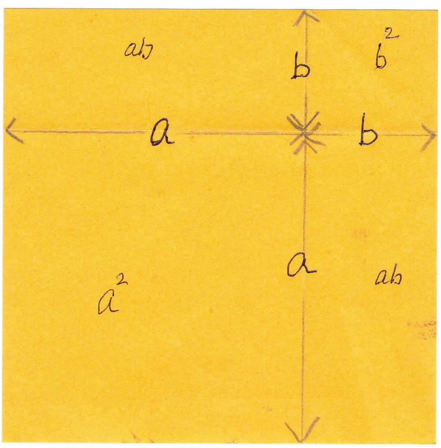

Figure 3

Ask students to take a square paper and fold them along the two indicated lines.

The two different lengths can be labelled as *a* and *b* as shown in Figure 3.

- What is the side of the original square? $a + b$
- What is the area of the original square? $(a + b)^2$
- What is the area of the big square? a^2
- What is the area of the small square? b^2
- What is the area of each rectangle? ab, ab
- What do they all sum up to? $a^2 + 2ab + b^2$

Hence $(a + b)^2 = a^2 + 2ab + b^2$

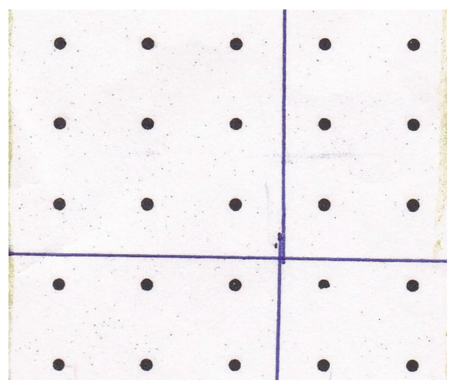

Figure 4

The teacher can also show this on a square dot paper as shown in Figure 4.

$$(3 + 2)(3 + 2) = 3 \times 3 + 3 \times 2 + 3 \times 2 + 2 \times 2$$

which is $(3 + 2)^2 = 3^2 + 2 \times 3 \times 2 + 2^2$

Again if $a = 3$ and $b = 2$ then $(a + b)^2 = a^2 + 2ab + b^2$

ACTIVITY 12

Objective: To show the identity $(a - b)^2$

This method works for positive numbers a and b, with $b < a$.

Ask students to take a square paper and fold them along the two indicated lines.

The two different lengths can be labelled as a and b as shown in Figure 5.

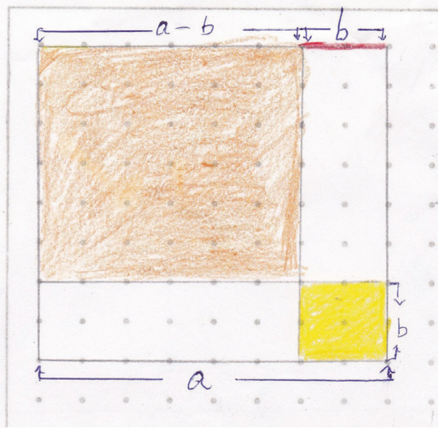

Figure 5

- What is the side of the original square? a
- What is the area of the original square? a^2
- What is the length of the portion which is being cut? b
- What is the length of the remaining part of the line? $a - b$
- What is the area of the square brown portion?

Each of its sides is $a - b$. Area of the brown square is $(a - b)^2$

- What is the area of the small yellow square? b^2
- What is the area of each outlined rectangle? ab
- Is it possible to remove both these two rectangles (of size ab) ?

Removal of two such rectangles will mean that b^2 will end up being removed twice.

In order to compensate for that we need to put back one b^2.

Hence $(a - b)^2 = a^2 - 2ab + b^2$

This can also be visualised as shown here.

×	a	$-b$
a	a^2	$-ab$
$-b$	$-ab$	b^2

ACTIVITY 13

Objective: To show the identity $a^2 - b^2 = (a + b)(a - b)$

This method works for positive numbers *a* and *b*, with $b < a$.

Ask students to take a square paper and fold them along the two indicated lines.

The two different lengths can be labelled as *a* and *b* as shown in Figure 6.

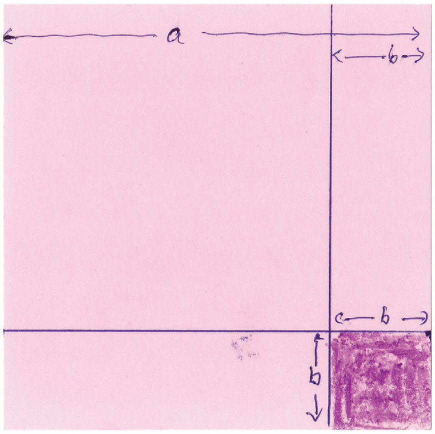

Figure 6

- What is the side of the original square? *a*
- What is the area of the original square? a^2
- What is the length of the portion which is being cut? *b*
- What is the area of the square purple portion? b^2

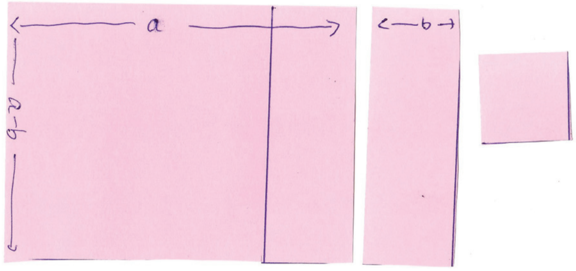

Figure 7

The remaining portions can be rearranged as shown in Figure 7.

- What is the length of the rearranged rectangle? $a + b$
- What is the breadth of the rearranged rectangle? $a - b$
- What is the area of this rearranged rectangle? $(a + b)(a - b)$

Hence $a^2 - b^2 = (a + b)(a - b)$

For further reading

Indices:

- http://www.teachersofindia.org/en/video/cloth-clips-powers-two
- http://www.teachersofindia.org/en/video/straw-powers-3
- http://www.teachersofindia.org/en/video/straw-powers-5
- http://www.teachersofindia.org/en/video/powers-2-3-and-5
- http://www.teachersofindia.org/en/video/exponential-identities
- https://www.youtube.com/watch?v=0fKBhvDjuy0

Identities:

- http://teachersofindia.org/en/presentation/algebraic-identities-visualized-one-more-time

With algebra tiles:

- http://teachersofindia.org/en/presentation/visual-proof-ab2-2ab-0
- http://teachersofindia.org/en/presentation/visual-proof-a-minus-b-whole-square
- http://www.teachersofindia.org/en/presentation/make-algebraic-proofs-visual-treat

Padmapriya Shirali

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