Problems for the MIDDLE SCHOOL

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Problem-IX-2-M-1. All prime numbers except 2 are odd numbers. Two consecutive odd numbers can both be prime. Such a pair is termed 'twin primes.' Except for the set (3,5,7), three consecutive odd numbers cannot be prime, since one of them must be a multiple of 3.

Two sets of twin primes can occur separated by a composite odd number. That is, out of five consecutive odd numbers, suppose all except the middle one are prime. [That is, if we denote the middle number by x, then $x - 4$, $x - 2$, $x + 2$ and $x + 4$ are all primes.] Such a set of prime numbers is termed a prime quadruple. Show that, with the exception of the case (5,7,11,13), x must then be a multiple of 15. (The next few sets that do follow the above rule are (11,13,17,19), (101,103,107,109) and (191,193,197,199).)

Problem-IX-2-M-2. Going a step further from the situation of the earlier problem, one can find a string of 9 consecutive odd numbers, out of which 6 are prime (such a set is termed a prime sextuplet), in the pattern described below. [If x is again taken to be the middle member (which is composite), then $x - 8$, $x - 4$, $x - 2$, $x + 2$, $x + 4$ and $x + 8$ are all prime.] The first such example is (7,11,13,17,19,23). Show that in all such instances except for the above, x must be a multiple of 105. The next two such sextuplets are (97,101,103,107,109,113) and (16057,16061,16063,16067,16069,16073); they do follow this rule.

Problem-IX-2-M-3. Suppose that the GCD of two numbers a and b is 10 and their LCM is 4200 (we write this as GCD $(a,b) = 10$ and LCM $(a,b) = 4200$) , what are the possible values of a and b?

Keywords: Prime numbers, LCM, GCD

Problem-IX-2-M-4. Given the three positive integers 390, 462 and 1190, find:

i. their pairwise LCM's
ii. their pairwise GCD's
iii. the LCM of all three
iv. the GCD of all three.

Now show that the following relations hold for these numbers:

A. $$\frac{\textit{Product of pairwise LCM's}}{\textit{Product of pairwise GCD's}} = \left\{ \frac{\textit{LCM of all three}}{\textit{GCD of all three}} \right\}^2$$

B. $$\textit{Product of pairwise GCD's} \times \left\{ \frac{\textit{LCM of all three}}{\textit{GCD of all three}} \right\} = \textit{product of given numbers}$$

C. *GCD of the pairwise LCM's = LCM of the pairwise GCD's.*

Solutions

Pedagogical Note: As a certain amount of algebra is unavoidable when discussing the general case, we use it as sparingly as possible in the following explanations and highlight it in blue font. However, the teacher is advised to use as many numerical and visual examples as possible and to refer to the general case only when the student is absolutely sure of the logic being followed.

1. We are given five consecutive odd numbers, all greater than 5, of which the first two are prime numbers, the last two are prime numbers, and the number in the middle is composite. We must show that this middle number is a multiple of 15.

We will use these two facts: (i) out of three consecutive odd numbers, one and only one is a multiple of 3; (ii) out of five consecutive odd numbers, one and only one is a multiple of 5.

Call the five consecutive odd numbers A, B, C, D, E. We are told that A and B are prime numbers, and so are D and E. We must show that the middle number C is a multiple of 15.

Among the three consecutive odd numbers A, B, C, one and only one is a multiple of 3. This can only be C, as A and B are prime numbers.

Among the five consecutive odd numbers A, B, C, D, E, one and only one is a multiple of 5. This can only be C, as A, B, D and E are prime numbers.

This means that C is a multiple of 3 and also a multiple of 5. Therefore, it is a multiple of 15.

2. We have just shown that x is a multiple of 15. It is sufficient if we show that x is a multiple of 7. One and only one out of seven consecutive odd numbers must be a multiple of 7. So if we took a prime sextuplet as described and x is not a multiple of 7, then one of the other odd numbers must be a multiple of 7. You can try it as above. Suppose the centre number is not a multiple of 7 and one of the other numbers is. As one out of 7 consecutive odd numbers must be a multiple of 7, either $x - 6$ or $x + 6$ must be a multiple of 7. [If $x - 6$ is a multiple of 7, then $x + 8$ differs from $x - 6$ by 14, so $x + 8$ must be a multiple of 7. Similarly for $x - 8$ and $x + 6$ which differ by 14. As this contradicts what is given, we can say that x must be a multiple of 7, and thereby of 105.]

3. For the given numbers, there are 8 solutions:

(10,4200), (40,1050), (50,840), (30,1400), (70,600), (200,210), (120,350) and (280,150).

The general case can be argued as follows:

We can take the numbers a and b to be of the form $a = pq$ and $b = pr$, with q and r mutually prime. Then GCD$(a, b) = p$ and LCM$(a, b) = pqr$. We also have $\frac{LCM\ (a,\ b)}{GCD\ (a,\ b)} = qr$. Both a and b are multiples of p, the GCD. So we need to partition the prime factors of qr into two sets with no common factor, in as many ways as possible. Multiplying p with two complementary sets of prime factors of qr would yield a possible solution to (a, b). Students may be guided to observe that in the general case the number of solutions is given by 2^{d-1}, d being the number of *distinct* prime factors of qr.

4. The pairwise LCM's are 30030, 46410, and 39270. The pairwise GCD's are 6, 10, and 14. The LCM of all three numbers is 510510, while their GCD is 2. In relation A, both sides of the equation would equal 65,155,115,025. In relation B, both sides would equal 214,414,200. In relation C, both sides would equal 210.

Addendum: To prove that this is true in the general case, we note that three numbers could have (a) one common factor, (b) pairwise common factors and (c) factors unique to each of the given integers. So we could assume the numbers to be P = $ab_1b_2c_1$, Q = $ab_2b_3c_2$ and R = $ab_3b_1c_3$. From this we obtain the following:

$LCM_{PQ} = ab_1b_2b_3c_1c_2$
$LCM_{QR} = ab_1b_2b_3c_2c_3$
$LCM_{RP} = ab_1b_2b_3c_3c_1$

$GCD_{PQ} = ab_2$
$GCD_{QR} = ab_3$
$GCD_{RP} = ab_1$

$LCM_{PQR} = ab_1b_2b_3c_1c_2c_3$
$GCD_{PQR} = a$.

Proceeding from this it should be easy to show that the above relations hold true in the general case.