Problems for the Senior School

Problem Editors: PRITHWIJIT DE & SHAILESH SHIRALI

Problem IX-2-S.1

The numbers 4 and 52 share the following features: both are sums of two squares; both exceed another square by 3. Thus:

$$4 = 0^2 + 2^2, \qquad 4 - 3 = 1^2;$$
$$52 = 4^2 + 6^2, \qquad 52 - 3 = 7^2.$$

Show that there are infinitely many numbers that have these two characteristics. [CRUX]

Problem IX-2-S.2

Let $f(n) = 25^n - 72n - 1$. Determine, with proof, the largest integer M such that $f(n)$ is divisible by M for every positive integer n. [CRUX]

Problem IX-2-S.3

Nine (not necessarily distinct) 9-digit numbers are formed using each digit 1 through 9 exactly once. What is the maximum possible number of zeros that the sum of these nine numbers can end with? [Kvant]

Problem IX-2-S.4

Note that $\sqrt{2\frac{2}{3}} = 2\sqrt{\frac{2}{3}}$. Determine conditions for which $\sqrt{a\frac{b}{c}} = a\sqrt{\frac{b}{c}}$, where a, b, c are positive integers. [CRUX]

Problem IX-2-S.5

Find all positive integers n satisfying the following condition: numbers $1, 2, 3, \ldots, 2n$ can be split into pairs such that if the numbers in each pair are added, and the sums are then multiplied together, the result is a perfect square. [Tournament of Towns]

Keywords: Sums of two squares, divisible

Solutions of Problems in Issue-IX-1 (March 2020)

Solution to problem IX-1-S.1

The midpoints of two sides of a triangle are marked. How can the midpoint of the third side be found using only a pencil and a straightedge?

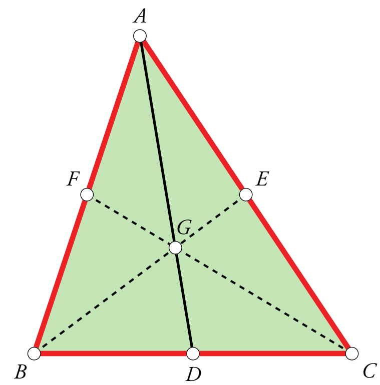

Figure 1.

Let us name the triangle ABC and let E and F be the midpoints of AC and AB, respectively (Figure 1). Join BE and CF and suppose G is their point of intersection. Since BE and CF are medians, G is the centroid of ABC. Join A and G and extend it to meet BC in D. AD is the median from A onto BC and therefore D is the midpoint of BC.

Solution to problem IX-1-S.2

Is it possible to cut several circles out of a square of side 10 *cm, so that the sum of the diameters of the circles is* 5 *metres or more?*

Yes, it is possible. Divide the square into 1 mm $\times$1 mm squares by drawing lines parallel to the adjacent edges of the given square. Inscribe circles into each of these $100 \times 100 = 10000$ squares. The sum of the diameters of all these circles is 10000 mm, i.e., 10 metres. Now cut them!!

Solution to problem IX-1-S.3

Suppose in a given collection of 2020 *integers, the sum of every* 100 *of them is positive. Is it true that the sum of all the* 2020 *integers is necessarily positive?*

Yes. To see this note that there cannot be more than 99 non-positive integers in the given collection. Otherwise the sum of some 100 of them will be non-positive contrary to the assumption. If $x_1, x_2, \ldots, x_k$ are the non-positive integers for some $k \leq 99$, then, by assumption,

$$(x_1 + x_2 + \cdots + x_k) + (x_{k+1} + x_{k+2} + \cdots + x_{100}) > 0.$$

The remaining integers (i.e., $x_{101}, x_{102}, \ldots$) are anyway positive, so the sum of all the 2020 integers is positive.

Solution to problem IX-1-S.4

Suppose integers a, b and c are such that $ax^2 + bx + c$ *is divisible by* 5 *for any integer x. Prove that each of a, b and c is divisible by* 5.

Setting $x = 0, 1, -1$ in succession, we find that c, $a + b + c$ and $a - b + c$ are divisible by 5. Therefore $(a + b + c) - (a - b + c) = 2b$ is divisible by 5, and since 2 and 5 have no factor in common, b is divisible by 5. Hence $a + c = (a + b + c) - b$ is divisible by 5. Hence $a = (a + c) - c$ too is divisible by 5. Hence a, b and c are all divisible by 5.

Solution to problem IX-1-S.5

The altitude dropped from A to BC in triangle ABC is not shorter than BC, and the altitude dropped from B to AC is not shorter than AC. Find the angles of triangle ABC.

Let X and Y be the feet of the altitudes dropped from A and B to BC and AC, respectively. Then $AX \geq BC$ and $BY \geq AC$. Also since AX and BY are altitudes, $AX \leq AC$ and $BY \leq BC$. Thus

$$AC + BC \leq AX + BY \leq AC + BC,$$

implying

$$AX + BY = AC + BC,$$

and therefore $AX = BC$, $BY = AC$. But then $BY = AC \geq AX$ and $AX = BC \geq BY$ implying $AX = BY$. Therefore:

$$AX = AC = BC = BY,$$

implying $X = Y = C$, i.e., X, Y, C coincide. Hence $\measuredangle C = 90°$ and $\measuredangle A = \measuredangle B = 45°$.