Problem Corner

Analysis of the Angle Trisection Procedure

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In the article by Shri Manoranjan Ghoshal which appeared in At Right Angles, November 2021, it is mentioned that approximate procedures for trisection of arbitrary angles exist, and one such procedure (elegant and simple to execute) was presented by the author. In this article, we analyse this procedure.

Proposed approximate procedure for trisection of angle

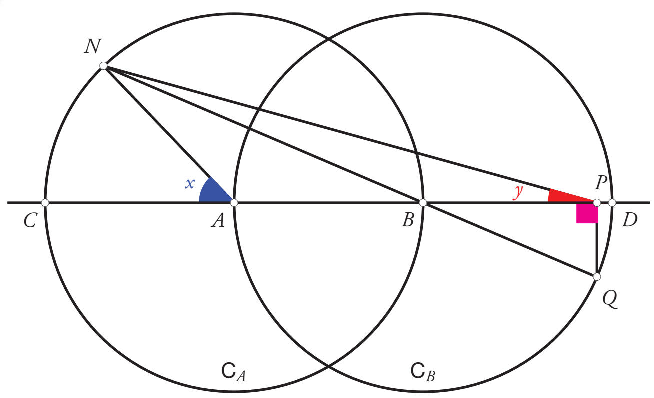

Figure 1.

- Let the angle to be trisected be denoted by x.
- Mark any two points A and B, 1 unit apart. Draw circles C_A and C_B centred at A and B respectively, with radius 1 unit each. Join AB and extend it to intersect C_A and C_B again at points C and D.

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- Locate a point N on C_A such that $\measuredangle NAC$ is equal to x (the angle to be trisected).
- Join NB and extend it beyond B till it meets C_B again at point Q.
- Draw QP perpendicular to line CD (with P on CD). Join NP.
- Then it will be found that $\measuredangle NPC \approx \frac{1}{3}\measuredangle NAC$, i.e., $y \approx \frac{1}{3}x$.

Analysis of the procedure

The task before us is to explain why this method gives such good results. We start by finding an expression for y in terms of x. For convenience, we have redrawn Figure 1.

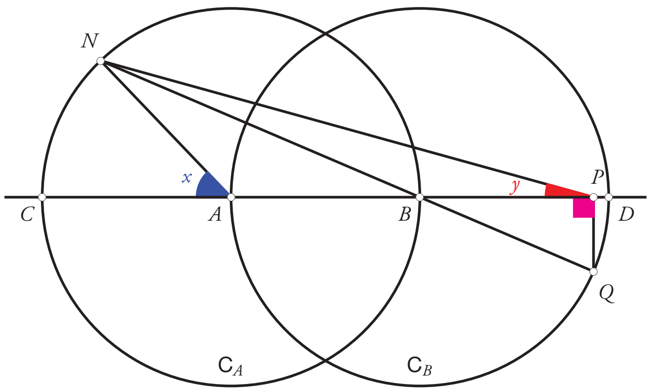


Figure 2.

We now make use of straightforward trigonometry. We have:

$$\measuredangle NBA = \frac{x}{2} = \measuredangle PBQ, \quad \therefore \; PQ = \sin\frac{x}{2}.$$

Next,

$$NB = 2\cos\frac{x}{2}, \qquad \therefore \; NQ = 1 + 2\cos\frac{x}{2}.$$

Also, $\measuredangle NQP = 90^\circ - x/2$, so by the sine rule,

$$\frac{NQ}{\sin(90^\circ + y)} = \frac{PQ}{\sin(x/2 - y)},$$

$$\therefore \; \frac{\sin(x/2 - y)}{\cos y} = \frac{\sin x/2}{1 + 2\cdot\cos x/2}.$$

Expanding the sine term on the left side, we get:

$$\frac{\sin x/2\,\cos y - \cos x/2\,\sin y}{\cos y} = \frac{\sin x/2}{1 + 2\cdot\cos x/2}.$$

This yields:

$$\sin\frac{x}{2} - \cos\frac{x}{2}\tan y = \frac{\sin x/2}{1+2\cdot\cos x/2},$$

$$\therefore\quad \cos\frac{x}{2}\tan y = \sin\frac{x}{2} - \frac{\sin x/2}{1+2\cdot\cos x/2} = \frac{2\cdot\sin x/2\cos x/2}{1+2\cdot\cos x/2},$$

$$\therefore\quad \tan y = \frac{2\sin x/2}{1+2\cdot\cos x/2},$$

so:

$$\boxed{y = \tan^{-1}\left(\frac{2\cdot\sin x/2}{1+2\cdot\cos x/2}\right).}$$

We have thus been able to express y in terms of x, explicitly. (It does not seem possible to simplify the expression further.)

The above expression allows us to compute y for any given value of x. For example, if $x = 60°$, we get:

$$y = \tan^{-1}\frac{1}{1+\sqrt{3}} = \tan^{-1}\frac{\sqrt{3}-1}{2} \approx 20.104°,$$

which is not too bad. The following table shows the closeness of the approximation for select angles:

x	15°	30°	45°	60°	75°	90°
y	5.002°	10.013°	15.043°	20.104°	25.206°	30.361°

We see that the relative error is significantly smaller when x is small; also that the trisected angle in each case is slightly *larger* than it should be.

A more precise error analysis

To obtain a precise estimate of the error and also explain the two observations made above, we shall express y as a power series in x. We make use of the following known power series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad \text{(valid for all real } x\text{)},$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \qquad \text{(valid for all real } x\text{)},$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \qquad \text{(valid for } -1 < x < 1\text{)}.$$

Using these results, we obtain:

$$2\cdot\sin\frac{x}{2} = 2\left(\frac{x}{2} - \frac{1}{3!}\cdot\frac{x^3}{2^3} + \frac{1}{5!}\cdot\frac{x^5}{2^5} - \cdots\right)$$
$$= x - \frac{x^3}{24} + \frac{x^5}{1920} - \cdots.$$

Next,

$$1+2\cdot\cos\frac{x}{2} = 1+2\left(1-\frac{1}{2!}\cdot\frac{x^2}{2^2}+\frac{1}{4!}\cdot\frac{x^4}{2^4}-\cdots\right)$$
$$= 3\left(1-\frac{x^2}{12}+\frac{x^4}{576}-\cdots\right),$$

so

$$\frac{1}{1+2\cdot\cos x/2} = \frac{1}{3}\left(1-\frac{x^2}{12}+\frac{x^4}{576}-\cdots\right)^{-1}$$
$$= \frac{1}{3}\left(1+\frac{x^2}{12}+\frac{x^4}{192}+\cdots\right).$$

The last step shown above may not look entirely clear, but we need to use the well-known expansion $(1-t)^{-1} = 1+t+t^2+t^3+\cdots$, which is valid for $-1<t<1$. By substituting the appropriate expression for t and ploughing through a lot of algebra, we obtain the stated result. Note that at each stage of the simplification, we make sure that we never carry with us terms beyond x^5.

From the above we obtain:

$$\frac{2\cdot\sin x/2}{1+2\cdot\cos x/2} = \frac{1}{3}\left(x-\frac{x^3}{24}+\frac{x^5}{1920}-\cdots\right)\cdot\left(1+\frac{x^2}{12}+\frac{x^4}{192}+\cdots\right)$$
$$= \frac{x}{3}+\frac{x^3}{72}+\frac{13x^5}{17280}+\cdots.$$

Hence we obtain:

$$\tan^{-1}\left(\frac{2\cdot\sin x/2}{1+2\cdot\cos x/2}\right) = \left(\frac{x}{3}+\frac{x^3}{72}+\frac{13x^5}{17280}+\cdots\right)-\frac{1}{3}\left(\frac{x}{3}+\frac{x^3}{72}+\frac{13x^5}{17280}+\cdots\right)^3+\cdots$$
$$= \frac{x}{3}+\frac{x^3}{648}+\frac{x^5}{31104}+\cdots,$$

after a substantial amount of simplification. This means that

$$y = \frac{x}{3}+\frac{x^3}{648}+\frac{x^5}{31104}+\cdots.$$

We deduce the following from the above relation:

- If x is very small, then the terms involving x^3 and higher powers may be neglected, and we have

  $$y \approx \frac{x}{3},$$

  as it should be. Figure 3 shows a graph of y against x. Observe the (extremely surprising) fact that for the portion $-1<x<1$, the graph almost exactly coincides with the graph of $y = x/3$. A graph of the same function is shown over a wider domain (Figure 4); we see some signs of non-linearity towards the edges.

- From the expression obtained for y, we see that y always *exceeds* $x/3$. This agrees with our earlier computations.

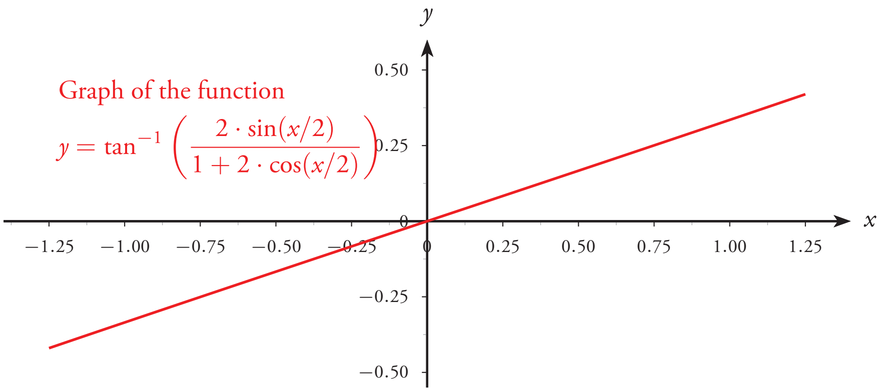


Figure 3.

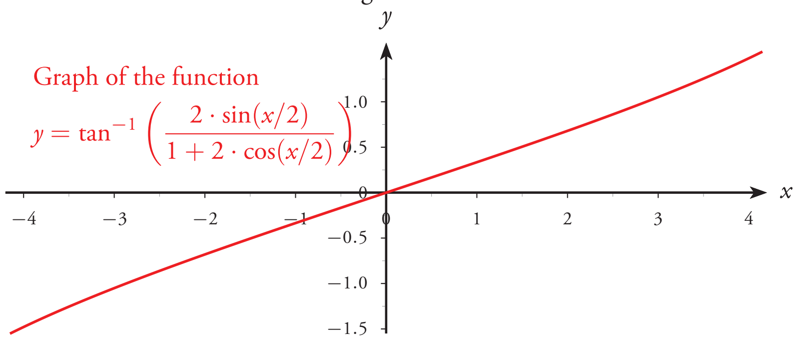


Figure 4.

- The relative error in taking y to be equal to $x/3$ is

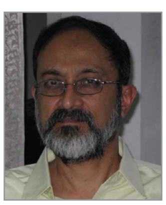
$$\frac{x^3/648 + x^5/31104 + \cdots}{x/3} = \frac{x^2}{216} + \frac{x^4}{10368} \approx \frac{x^2}{216}.$$

  So the percentage error in taking y to be equal to $x/3$ is about $x^2/2.16$, i.e., roughly $x^2/2$.

- The above finding explains why the accuracy of the procedure is so much better when the angle is small, and why the accuracy goes down as the angle increases in size.

- If the angle to be trisected is close to 1 radian, then the percentage error according to the above analysis should be roughly 0.5. This agrees with the computation when the angle to be trisected is 60°. We had earlier found the error to be 0.104. So the percentage error is $0.104/20 \times 100$, which is close to 0.5, just as expected.

Concluding remark. To come across an approximate angle trisection procedure of such a simple nature (i.e., where the number of steps in the construction is relatively small) and which at the same time gives such accurate results (at least for small angles), is quite surprising. It just shows that there are unexpected but most pleasing surprises to be found in mathematics at every level!



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