The Painter's Paradox

Comparative Analysis of Gabriel's Horn and Triangular Pipe with Koch's Fractal Shaped Cross Section

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Abstract

This paper presents the Painter's Paradox—a highly counterintuitive situation where a painter is able to fill a certain 3-dimensional object with paint but is unable to fully paint the surface of that object.

Mathematically, this paradox illustrates that a 3-dimensional object can have a finite volume while having an infinite surface area.

A well-known object like Gabriel's Horn is a classic example used to illustrate this paradox. To study it, we require a basic understanding of integral calculus and the concepts of surface area and volume. However, one can construct other objects that illustrate the same paradox, using only high school geometry and geometric series.

At the heart of this paradox lies the counterintuitive nature of infinite series.

The Study

The "**Painter's Paradox**" is presented in two sections.

Section 1 uses the 'Gabriel Horn' to illustrate the paradox; it assumes the reader's acquaintance with integral calculus and concept of a solid of revolution—in this case, the solid obtained by revolving the region bounded by the function $f(x) = 1/x$ and the x-axis around the x-axis.

Keywords: Paradox, Geometric progression, infinite series

Section 2 illustrates similar results using a new 3-D object based on fractals and requires only the reader's familiarity with high school mathematics.

Section 1. Gabriel's Horn

Figure 1 shows the plot of function y = 1/x for x $\geq$ 1. The 3D region is generated by rotating the curve by 2"πradians about the *x*-axis to generate the object known as Gabriel's Horn with volume V and surface area S.

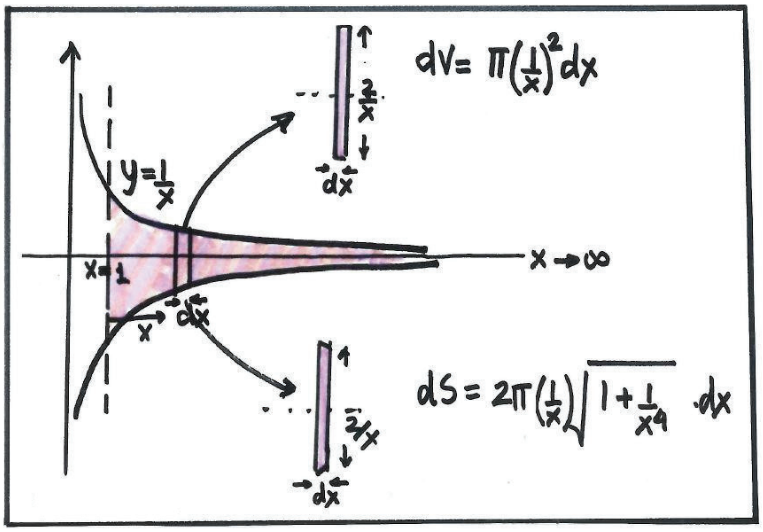

Figure 1.

Volume (dV) of a thin circular disk of thickness d*x* at distance *x* from the origin is given by:

$$dV = \pi y^2 dx \text{ where } y = \frac{1}{x} \text{ which gives } dV = \frac{\pi}{x^2} dx.$$

Volume (V) of the Gabriel Horn is obtained by integration as shown below:

$$V = \int_0^V dV = \int_1^\infty \frac{\pi}{x^2} dx = \pi\left[-\frac{1}{x}\right]_1^\infty = \pi\left[-\frac{1}{\infty} + \frac{1}{1}\right] = \pi.$$

Surface area (dS) of a thin circular disk at distance x from origin of thickness dx is given by:

$$dS = 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \text{ where } y = \frac{1}{x} \text{ which gives } \frac{dy}{dx} = -\frac{1}{x^2}$$

Hence:

$$S = \int_0^S dS = \int_1^\infty 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_1^\infty \frac{2\pi}{x}\sqrt{1+\left(-\frac{1}{x^2}\right)^2}\, dx$$

So:

$$S = \int_1^\infty \frac{2\pi}{x^3}\sqrt{x^4+1}\, dx$$

Therefore $S \int_1^\infty \frac{2\pi}{x^3}\sqrt{x^4} dx = \int_1^\infty \frac{2\pi}{x} dx = 2\pi\,[\ln x]_1^\infty$, which is not finite, since $\ln x \to \infty$ when $x \to \infty$.

Thus the surface area of Gabriel Horn is infinite while its volume is finite, i.e., π.

Section 2. Triangular Pipe with Koch's fractal shaped cross section

Figure 2 shows the cross section of a triangular pipe of unit height (i.e. h = 1) which undergoes repeated iterations following Koch's curve rule.

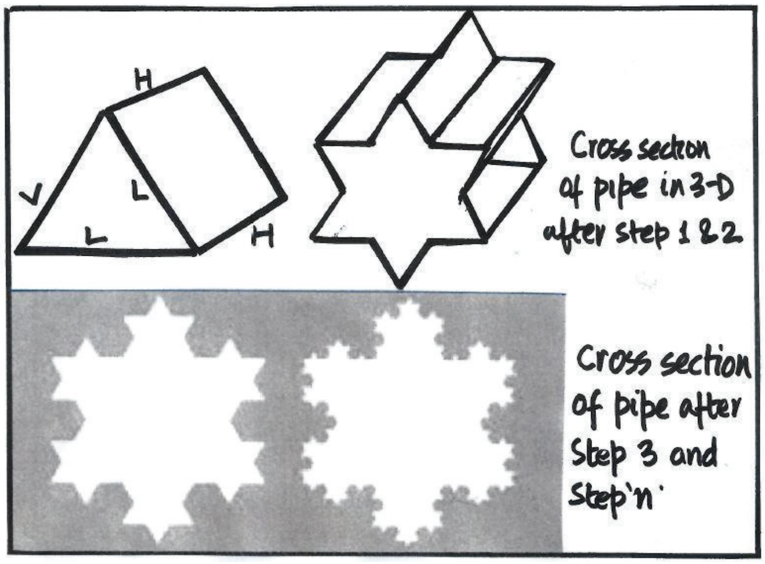

Figure 2.

Table 1 and Table 2 give the explicit formulae for volume and surface area of the object shown in Figure 2 after n iterations.

Hence as $n \to \infty$, we also have $S \to \infty$.

Thus the surface of a triangular pipe whose cross-section is shaped like Koch's fractal has infinite area while its volume is bounded to 9/5$^{\text{th}}$ of its original volume.

Iteration	Number of new added △	Area of each new △	Total Volume (fig 2) (V = A × h = A × 1)
0	0	0	0
1	1	A	V
2	4	$\frac{A}{9}$	$V = V + \frac{4V}{9} = \frac{13V}{9}$
3	16	$\frac{A}{81}$	$V = V + \frac{4V}{9} + \frac{16V}{81} = \frac{133V}{81}$
4	64	$\frac{A}{729}$	$V = V + \frac{4V}{9} + \frac{16V}{81} + \frac{64V}{729} = \frac{1261V}{729}$
n	4^{n-1}	$\frac{A}{9^{n-1}}$	$V = V + \frac{4V}{9} + \frac{16V}{81} + .. + \frac{4^{n-1}V}{9^{n-1}}$
∞	∞	0	$V = \sum_{n=1}^{\infty} \frac{4^{n-1}V}{9^{n-1}} = \frac{9V}{5}$

Table 1.

Iteration	Number of Segments	Length (L) of each segment	Perimeter (fig 2) $P = 3 \times L$	Total Surface area (fig 2) $(S = P \times h = P \times 1)$
0	3	1	3	3
1	12	$\frac{1}{3}$	4	4
2	48	$\frac{1}{9}$	$\frac{16}{3}$	$S = \frac{16}{3}$
3	192	$\frac{1}{27}$	$\frac{64}{9}$	$S = \frac{64}{9}$
4	768	$\frac{1}{81}$	$\frac{256}{27}$	$S = \frac{256}{27}$
n	34^n	$\frac{1}{3^n}$	$\frac{4^n}{3^{n-1}}$	$S = \frac{4^n}{3^{n-1}}$

Table 2.

Section 3. Conclusion

The investigation of volume and surface area of two different objects using calculus and high school mathematics provides an interesting glimpse of the Painter's Paradox.

Further study can be carried out to investigate the logical fallacy of the statement that if we can fill the Gabriel's Horn with a finite amount of paint, then the inner surface area is automatically painted completely.

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