Triangles and Pell's Equation

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In a WhatsApp group of Math enthusiasts, a question was posed some time ago: 13,14,15 are the sides of a triangle with rational area and side lengths that are consecutive integers. Can we find more such triangles? This question led me to ask, how many such triangles exist? Can we come up with a general formula to generate such triangles? In this article, I explore these questions further.

Problem. Find all triangles with sides as consecutive integers and rational area.

Solution. Let the side lengths of the triangle be $x-1, x, x+1$, where x is a positive integer, $x > 1$.

Such a triangle exists if and only if the triangle inequality is obeyed, which implies here that $(x-1)+x > x+1$. This reduces to $x > 2$.

Next, we find the area of this triangle using Heron's formula:

$$A = \frac{1}{4}\sqrt{(3x)(x+2)(x)(x-2)} = \frac{x}{4}\sqrt{3\left(x^2-4\right)}.$$

This shows that the area of the triangle is rational if and only if $3\left(x^2-4\right)$ is the square of a rational number. Since $3\left(x^2-4\right)$ is an integer, this means that it is a perfect square, say y^2. So we must look for positive integer pairs (x, y) such that $y^2 = 3\left(x^2-4\right)$.

We first show that x cannot be odd. For, if x were odd, then we would have $x^2 \equiv 1 \pmod 4$, and this would lead to

$$y^2 \equiv 3 \pmod 4$$

which is not possible, as no perfect square is of the form $3 \pmod 4$. Therefore x is even. But this implies that y too is even.

Let $x = 2a$ and $y = 2b$, where a and b are positive integers (with $a > 1$, since $x > 2$). Substituting, we get $4b^2 = 12a^2 - 12$, and therefore

$$b^2 = 3a^2 - 3.$$

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From this we deduce that b is a multiple of 3, say $b = 3c$, where c is a positive integer. Substituting, we get $9c^2 = 3a^2 - 3$, and therefore $3c^2 = a^2 - 1$, i.e.,

$$a^2 = 3c^2 + 1.$$

So the task reduces to finding positive integer pairs (a, c) such that $a^2 = 3c^2 + 1$. We have arrived at a well-studied kind of equation: the *Pell equation*.

The first solution in positive integers is $(a, c) = (2, 1)$. It is possible to show that the positive integer solutions are all given by $(a, c) = (a_k, c_k)$, where

$$a_k + c_k\sqrt{3} = \left(2 + \sqrt{3}\right)^k.$$

This means that the solutions are $(a, c) = (2, 1), (7, 4), (26, 15), \ldots$. To solve for a_k we have:

$$a_k + c_k\sqrt{3} = \left(2 + \sqrt{3}\right)^k,$$
$$a_k - c_k\sqrt{3} = \left(2 - \sqrt{3}\right)^k,$$

which yield

$$a_k = \frac{(2 + \sqrt{3})^k + (2 - \sqrt{3})^k}{2}.$$

Substituting $x = 2a_k$ gives:

$$x = \left(2 + \sqrt{3}\right)^k + \left(2 - \sqrt{3}\right)^k.$$

We thus get the solutions $x = 4, 14, 52, 194, 724, 2702, 10084, \ldots$.

These lead to the following triangles:

x	4	14	52	194	724	...
Sides	3, 4, 5	13, 14, 15	51, 52, 53	193, 194, 195	723, 724, 725	...
Area	6	84	1170	16296	226974	...

Can you see why all the areas are integers?

Closing comment. An interesting question to explore further would be to characterize all integer-sided triangles with sides in an arithmetic progression and rational area. (Hint: The idea is similar to the original question: the arithmetic progression constraint ensures that the expression that needs to be a perfect square is a quadratic in x.)

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