Student Corner

Connecting φ and π

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In this short article, we bring out an interesting connection between two fundamental constants of mathematics: π (which is sometimes known as "Archimedes's constant") and ϕ (the golden ratio). The connection comes about through consideration of an improper integral.

In integral calculus, integrals may be broadly classified into two categories: proper and improper. The term 'improper' means that the limits of the integral are not finite, i.e., one or both of the limits may be infinity (positive or negative). An integral having any one limit infinite can be expressed similarly as the following limit:

$$\int_0^\infty f(x)\,\mathrm{d}x = \lim_{n\to\infty}\int_0^n f(x)\,\mathrm{d}x. \qquad (1)$$

Here, we study a particular improper integral and discuss its solution, which has a nice - looking closed form expression in terms of known universal constants. In the end, we comment on the general form of the integral. The first crucial claim we make is as follows:

Lemma.

$$\lim_{x\to\infty} x\tan^{-1}\left(\frac{2}{x^2+1}\right) = 0.$$

Proof. We use the method of series expansion. We have,

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots,$$

so:

$$x\tan^{-1}\left(\frac{2}{x^2+1}\right) = x\left(\frac{2}{x^2+1}\right) - \frac{x}{3}\left(\frac{2}{x^2+1}\right)^3 + \frac{x}{5}\left(\frac{2}{x^2+1}\right)^5 - \cdots.$$

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Now $\frac{x}{x^2+1} \to 0$ as $x \to \infty$. Thus, defining $T(x,n) := \frac{x}{(x^2+1)^n}$, we see that

$$\lim_{x,n\to\infty} T(x,n) \;=\; 0, \quad \text{for } n = 1, 3, 5, \ldots.$$

Therefore $\lim_{x\to\infty} c\, T(x,n) \;=\; 0$, for any $c \;\in\; \mathbb{R}$.

Using the above, we see that the required limit, which is the sum of above similar terms, is 0. Thus

$$\lim_{x\to\infty} x \tan^{-1}\left(\frac{2}{x^2+1}\right) = 0.$$

Now, we come to our main problem.

Problem

Evaluate the integral

$$\int_0^\infty \tan^{-1}\left(\frac{2}{x^2+1}\right)\, \mathrm{d}x.$$

Solution

We use integration by parts, taking $\tan^{-1}\left(\frac{2}{x^2+1}\right)$ as the first function. Hence, denoting the given integral by I, we get :

$$\begin{aligned}
I &= \int_0^\infty \tan^{-1}\left(\frac{2}{x^2+1}\right) . \, 1 \, \mathrm{d}x \\
&= \left[x \tan^{-1}\left(\frac{2}{x^2+1}\right)\right]_0^\infty - \int_0^\infty x\left[\frac{d}{dx}\tan^{-1}\left(\frac{2}{x^2+1}\right)\right]\, \mathrm{d}x \\
&= \left[x \tan^{-1}\left(\frac{2}{x^2+1}\right)\right]_0^\infty - \int_0^\infty x\left(\frac{(x^2+1)^2}{x^4+2x^2+5}\cdot\frac{-4x}{(x^2+1)^2}\right)\, \mathrm{d}x \\
&= \lim_{x\to\infty} x\tan^{-1}\left(\frac{2}{x^2+1}\right) - 0 + \int_0^\infty x\left(\frac{(x^2+1)^2}{x^4+2x^2+5}\cdot\frac{4x}{(x^2+1)^2}\right)\, \mathrm{d}x \\
&= 0 + \int_0^\infty \frac{4x^2}{4+(x^2+1)^2}\, \mathrm{d}x \\
&= \int_0^\infty \frac{4x^2}{x^2\,(x^2+2+5/x^2)}\, \mathrm{d}x \;=\; \int_0^\infty \frac{4}{x^2\,(x^2+2+5/x^2)}\, \mathrm{d}x \\
&= \int_0^\infty \frac{4}{(x^2-2\sqrt{5}+5/x^2)+22\sqrt{5}}\, \mathrm{d}x \\
&= \int_0^\infty \frac{4}{\left(x-\sqrt{5}/x\right)^2+2+2\sqrt{5}}\, \mathrm{d}x.
\end{aligned}$$

A crucial step here is the introduction of the factor $2\sqrt{5}$.

Now we carry out this transformation: $x \mapsto x/\sqrt{5}$. The integral becomes

$$I = \int_{\infty}^{0} \frac{4}{\left(\sqrt{5}/x - x\right)^2 + 2 + 2\sqrt{5}} \cdot \left(\frac{-\sqrt{5}}{x^2}\right) \, \mathrm{d}x.$$

Taking into account the negative sign, the expression becomes

$$I = \int_{0}^{\infty} \frac{4}{\left(\sqrt{5}/x - x\right)^2 + 2 + 2\sqrt{5}} \cdot \left(\frac{\sqrt{5}}{x^2}\right) \, \mathrm{d}x.$$

So, we have:

$$\begin{aligned} I + I &= \int_{0}^{\infty} \frac{4}{\left(x - \sqrt{5}/x\right)^2 + 2 + 2\sqrt{5}} \, \mathrm{d}x + \int_{0}^{\infty} \frac{4}{\left(\sqrt{5}/x - x\right)^2 + 2 + 2\sqrt{5}} \cdot \left(\frac{\sqrt{5}}{x^2}\right) \, \mathrm{d}x \\ \implies 2I &= \int_{0}^{\infty} \frac{4}{\left(x - \sqrt{5}/x\right)^2 + 2(1+\sqrt{5})} \cdot \left(1 + \frac{\sqrt{5}}{x^2}\right) \, \mathrm{d}x \\ \implies I &= \int_{0}^{\infty} \frac{2}{\left(x - \sqrt{5}/x\right)^2 + 2(1+\sqrt{5})} \cdot \left(1 + \frac{\sqrt{5}}{x^2}\right) \, \mathrm{d}x. \end{aligned}$$

We need one more substitution. Putting $t = x - \sqrt{5}/x$, we get

$$I = \int_{-\infty}^{\infty} \frac{2}{t^2 + 2(1+\sqrt{5})} \, \mathrm{d}t.$$

It is well-known that $\phi = \dfrac{1+\sqrt{5}}{2}$ is the Golden Ratio (for more details, one can refer to [4]), a fundamental constant appearing in theory as well as the practical world, and having numerous beautiful properties. Observe that $2(1+\sqrt{5}) = 4\phi$. So,

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \frac{2}{t^2 + 2(1+\sqrt{5})} \, \mathrm{d}t = \int_{-\infty}^{\infty} \frac{2}{t^2 + 4\phi} \, \mathrm{d}t \\ \implies I &= \int_{-\infty}^{\infty} \frac{2}{t^2 + 4\phi} \, \mathrm{d}t = \frac{1}{\sqrt{\phi}} \tan^{-1}\left(\frac{t}{2\sqrt{\phi}}\right)\Bigg|_{-\infty}^{\infty} = \frac{1}{\sqrt{\phi}}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{\sqrt{\phi}}. \end{aligned}$$

Therefore, we have our answer:

$$\boxed{\int_{0}^{\infty} \tan^{-1}\left(\frac{2}{x^2+1}\right) \, \mathrm{d}x = \frac{\pi}{\sqrt{\phi}}}.$$

Closing remarks.

- We have used only "high-school techniques" in finding out the answer.
- I first saw this problem in a post by Prof. Brian Sittinger, in Quora.
- An aspect of this problem which needs to be mentioned is that the appearance of the golden ratio in the answer is due to the 2 in the numerator of the original integral. Thus, we would not be able to get this compact form had there been any other natural number in place of 2, i.e., if we had the 'general' form of the integral. Thus, the 2 plays a crucial role.

Readers interested in exploring more such beautiful integrals involving special functions and mathematical constants could refer to [1], [3], [5] and [6].

References

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