The Minimal Instruments of Geometry – I

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1. Introduction

Euclid's *Elements* (~300 BCE) built the edifice of (plane) Geometry using a toolkit comprising of two instruments: the 'straight edge' and the 'collapsible compass' [1]. Many centuries later (1941 CE), in *Basic Geometry*, George Birkhoff and Ralph Beatley provided an alternative construction of this edifice using a three-instrument toolkit which contemporary students continue to use: the 'ruler', the 'compass' and the 'protractor' [2]. In contrast, a few centuries before Euclid (~800 BCE), Indian vedic texts *(Shulbasutras)* recommended the 'rajju', i.e., a rope, as the lone instrument to be used for geometrical constructions [3].

In the first part of this two-part article, we introduce these three toolkits and explore some ideas for rope based geometrical constructions. In the second part of the article, we will compare the three toolkits in terms of their ability to do various geometric constructions and discuss some ideas to enable the construction of the tools themselves[1].

2. Euclid and Birkhoff-Beatley Toolkits

We begin by contrasting Birkhoff and Beatley's toolkit with that of Euclid.

2.1 Ruler vs. Straight Edge

A straight edge could be described as a ruler without markings (Figure 1). Thus, a straight edge and a ruler can both be used to draw a straight line but, unlike a ruler, a straight edge cannot quantify the length of that line.

1 Historically, things may have been done differently from the ideas discussed here. Our focus however, is on the mathematical correctness of these ideas.

Keywords: Euclid, plane geometry, instrument box, ruler, compass, protractor, rope

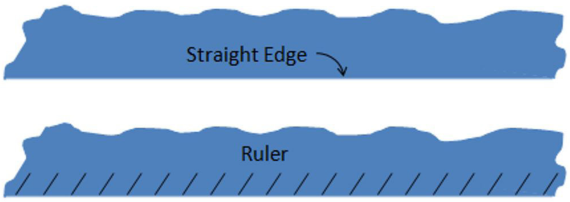

Figure 1: Straight edge (Euclid) vs. ruler (Birkhoff- Beatley)

Birkhoff and Beatley assumed that the ruler is marked with infinite resolution, so any length (rational or irrational) can be measured/constructed exactly using such a ruler.[2]

2.2 Normal vs. Collapsible Compass

The 'normal' compass is the one found in contemporary school geometry boxes. This compass 'remembers' the distance between its steady and adjustable legs when the 'hinge' is screwed tight (Figure 2).

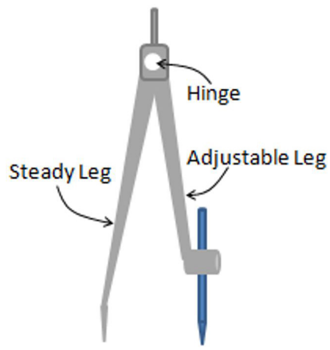

Figure 2: Basic parts of a 'normal' compass

If the hinge is loosened, the legs of the compass will 'collapse' together when they are lifted from the paper. In this case, the compass cannot 'remember' the distance between the two legs. This then becomes Euclid's collapsible compass. Euclid showed that the straight edge and collapsible compass can perform the job of a 'normal' compass in terms of reproducing a given length at a different location [1 pp. 244-246]. We therefore, use the word 'compass' to refer to a 'normal' compass hereafter.

2.3 Protractor

The protractor is used to construct and measure angles (Figure 3). This instrument from the Birkhoff-Beatley toolkit has no direct equivalent in Euclid's toolkit.

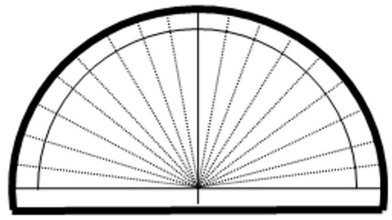

Figure 3: The protractor

Like the ruler, the protractor too is assumed to be marked with infinite resolution and can therefore be used to measure/construct any angle.

3. The Great Indian Rope Trick

Let us now see how the rope can be used as a geometrical instrument. In the *Shulbasutra* context, geometrical constructions were used to build life-size architectural structures with dimensions up to a few tens of metres. However, the geometrical principles involved in these constructions are also valid at smaller scales, just as those of Euclid or Birkhoff-Beatley are valid at scales larger than a sheet of paper.

3.1 Rope as a 'Markable' Straight Edge

A rope stretched taut forms a straight line. Thus, to construct a straight line between any two points on the ground, we simply stretch a rope taut between the two points and use it as a straight edge (Figure 4). Further, the distance between the two points can be marked by tying knots at the appropriate locations on the rope. In contrast, on Euclid's straight edge we are not allowed to make any markings to record a length.

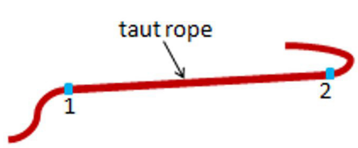

Figure 4: Rope as a straight edge between points 1 and 2

2 If you have a ruler marked only at integer lengths, you can only measure lengths such as 1 unit, 2 units, 3 units… Or, if you have a ruler marked only at half-integer lengths, you can only measure lengths such as 0.5 units, 1 unit, 1.5 units… B & B take their ruler to have infinite resolution. You can measure any length with it, rational or irrational.

We assume that our rope based geometrical constructions are being performed over 'level' ground. This is equivalent to assuming that the Euclid or the Birkhoff-Beatley toolkit based constructions are being performed on a 'plane' sheet of paper.

3.2 Rope as a Compass

Let us say that we are given two points. One point marks the desired centre of the circle while the distance between the points is the desired radius. We can now stretch out a rope between them and mark it corresponding to the two points. If the point of the rope corresponding to the centre of the circle is now used as a pivot as we walk around with the rope stretched taut along the ground, the other mark on the rope will describe a circle on the ground as shown in Figure 5.

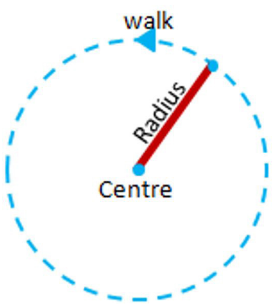

Figure 5: Using a rope to make a circle

Since the rope can be marked, it can 'remember' a distance and function as a 'normal' compass.

4. Unique Rope Constructions

The ability to use the rope as a straight edge and as a compass means that it is at least as good as Euclid's toolkit for geometrical constructions. Additionally, the flexibility of the rope, coupled with the ability to mark it, allows some alternatives to Euclidean constructions as well as some constructions that are impossible with the rigid instruments of the Euclid and Birkhoff-Beatley toolkits. We look at some such constructions here.

4.1 Multiples and Fractions of a Length

The *Shulbasutras* describe several constructions requiring the division of a finite straight line into a number of equal parts but do not explain how this division is to be done [3 pp. 41–42]. That may be because with a rope, both multiplication and division of a length by any natural number is a fairly intuitive construction.

Figure 6 illustrates the multiplication of a given length. If L be the length marked off by points '1' and '2' on the rope, then the points '1' and '3' will be $2L$ apart while the points '1' and '4' will be $3L$ apart when the rope is unfolded and stretched taut again.

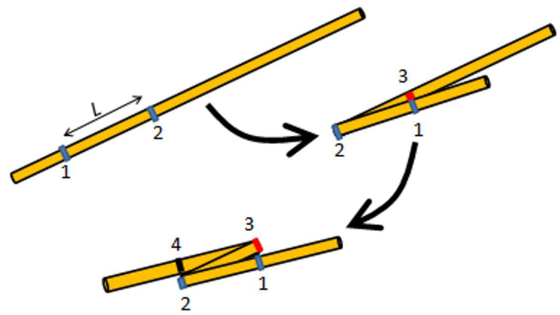

Figure 6: Multiplying a given length

Similarly, Figure 7 shows how a length can be divided. The division of L by 2 can be achieved in one shot by making the fold at '*a*' such that the marks '1' and '2' align, with each of the folded sections stretched taut. When the rope is unfolded and stretched out again, the distances from '1' to '*a*' and from '*a*' to '2' will each be $L/2$.

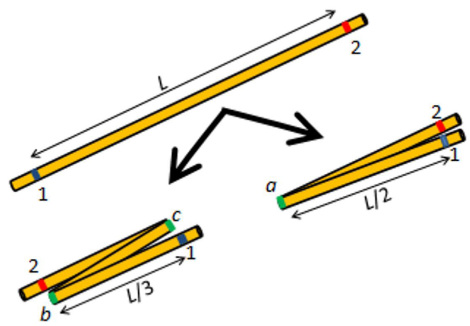

Figure 7: Dividing a given length

The division of L by 3 takes some 'tuning'. We need to make two folds between the '1' and '2' marks and adjust them such that one fold each aligns with these marks while keeping each folded section of the rope stretched taut. In Figure 7, these fold locations have been marked as '*b*' and '*c*'. Then, the distances '1' to '*b*' or '*b*' to '*c*' or '*c*' to '2' are each $L/3$.

In theory, this technique can be used to multiply or divide a length by any natural number N using N-1 folds. Consequently, any rational multiple of the length L can be obtained by this technique.

It is also possible to find the square root of a given length. This can be done by using the procedure set out in the *Shulbasutras* to construct a square that is equal in area to a given rectangle [3 pp. 83–85, 4]. Following this technique for a rectangle having the given length and unit breadth, we will get a square whose side is the square root of the given length.

4.2 Multiples and Fractions of an Angle

The rope can be used to measure curved lengths. In practice, the curves can be marked as a groove in the ground in which the rope can be fit and the length of the curve marked on it. This idea can be applied to a circle to convert a given angle θ° into an arc-length as shown in Figure 8 (using the property that angle (in radians) = arc/radius).

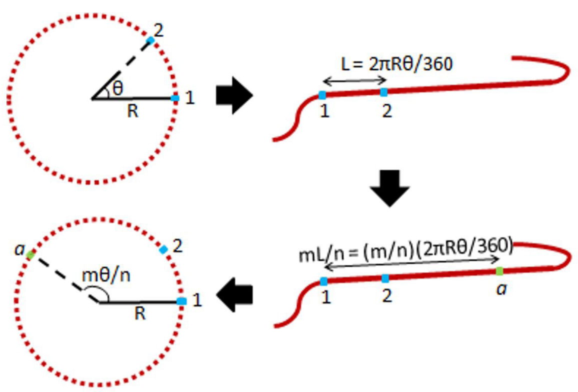

Figure 8: Constructing rational multiple of given angle

We use the vertex of the angle as the centre and the length of one of the arms forming the angle as radius to construct a circle around which we fit a rope. We mark the rope where the two arms of the angle (extended if required) cut the circle. This arc length can then be multiplied by any rational factor m/n to construct the angle $(m\theta°)/n$ (mod 360).

Apart from rational multiples, square root length constructions can also be used for corresponding irrational angle constructions.

4.3 Conic Sections

Though this has not been discussed in the *Shulbasutras* and is probably a later discovery, the rope enables the construction of ellipses, hyperbolas and parabolas [5]. This cannot be achieved by either the Euclid or the Birkhoff-Beatley toolkits.

Figure 9 illustrates the construction of an ellipse using a single rope F_1MF_2 of length R. We exploit the property of the ellipse that any point on it is such that the sum of its distance from two fixed points (the foci F_1 and F_2 of the ellipse) is some constant.

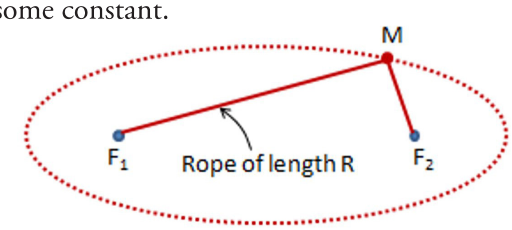

Figure 9: Construction of an ellipse

The marker M traces out the ellipse as it is moved around while keeping the two sections of the rope, F_1M and F_2M, stretched taut. Though the location of M along the length of the rope is not fixed, $F_1M + F_2M$ will always equal the total length R of the rope, which is a constant.

Figure 10 shows a hyperbola construction using ropes. A taut rope F_1P of length R is used as a straight edge pivoting at F_1 while another rope F_2MP of length L is used as a flexible rope which is folded into two sections of variable lengths PM and MF_2 by the marker M. The points F_1 and F_2 are fixed. The construction relies on the property of the hyperbola that all points on it are such that the difference in their distances from two fixed points (the foci F_1 and F_2) is some constant.

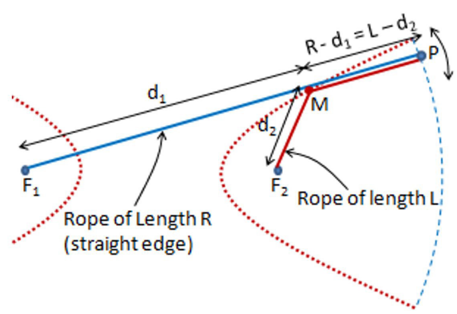

Figure 10: Construction of a hyperbola

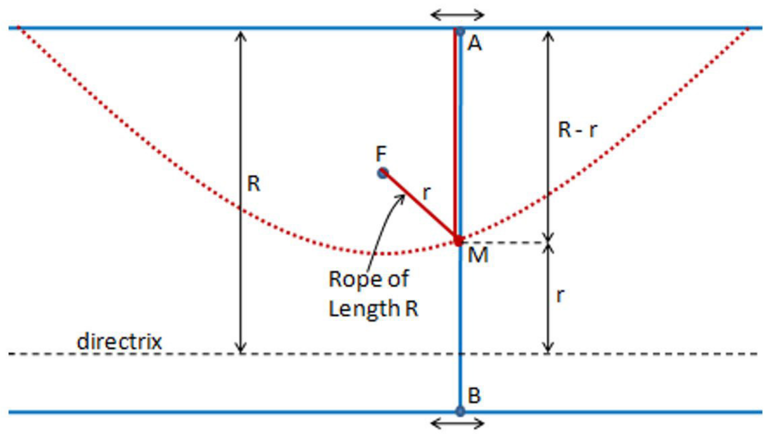

Figure 11: Construction of a parabola

As we can see in Figure 10, the length of the section PM of the flexible rope that lies along F_1P is $L - d_2$ where d_2 is the distance of the marker M from the focus F_2. Further, PM is also equal to $R - d_1$ where d_1 is the distance of M from the focus F_1. Thus, as F_1 P rotates, the position of M varies, but we always have $PM = L - d_2 = R - d_1$, i.e., $d_1 - d_2 = R - L$, a constant. This is exactly the condition that defines a hyperbola.

The construction needs to be mirrored to get both halves of the hyperbola. In practice, it is far more convenient to use a rigid straight edge instead of a rope for F_1P.

Lastly, Figure 11 shows the construction of a parabola. There are two parallel (horizontal) fixed ropes on which a third (vertical) movable rope AB gets pushed (right or left) by the marker M placed on the flexible rope FMA. For A and B to always lie on the fixed horizontal ropes, AB must move parallel to its own previous position. During this movement, the section AM of the flexible rope lying along AB will be of length $R - r$ where R is the total length of the flexible rope, and r is the distance of the marker M from the fixed point F, the focus of the parabola. The length $FM = r$ makes up the remaining part of the flexible rope.

We now define the 'directrix' as a straight line parallel to the two fixed horizontal ropes and at a distance R below the top horizontal rope. Then, the distance between M and the directrix must also be r. Thus, as the marker M pushes the rope AB to the right or left, it will describe the locus of points which are equidistant from F and the directrix, which is the definition of a parabola.

Again, in practice, the mechanical set-up is far more convenient if we use rigid materials for the fixed parallel ropes and movable vertical rope.

4.4 Practical considerations

The rope based geometrical constructions described here would ideally need ropes of zero thickness and infinite flexibility. In practice, therefore, these constructions will have inaccuracies. Also, as mentioned earlier, though the mathematical principles involved in these constructions are scalable, carrying out rope-based constructions for dimensions of a few centimetres or smaller would involve significant handling difficulties. For all such cases, a thread instead of a rope may work better. Another source of inaccuracies could be some inherent elasticity of the rope.

Having reviewed some uses of the rope as a geometrical instrument, we will analyze how it compares with the Euclid and Birkhoff-Beatley toolkits in the next part of the article.

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