Features

How Archimedes showed that pi is approximately 22 by 7

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Background

The mathematical constant π is defined as

$$\pi = \frac{C}{D}, \tag{1}$$

where C is the circumference and D is the diameter of a circle. The value of C/D is constant, regardless of the size of the circle. It is known that π is an irrational number, which means that it cannot be expressed as a ratio of two integers. Its value correct to the sixth decimal place is 3.141592 and fractions such as 22/7 and 355/113 are often used to approximate it. Since Archimedes was one of the first persons (around 200 BCE) to suggest a rational approximation of 22/7 for π, the number is sometimes referred to as *Archimedes' constant*. In 2019, Emma Haruka Iwao from Japan numerically computed about 31 trillion digits of π, and the record for memorizing the maximum number of digits (71,000) is held by Rajveer Meena from India.

In the first part of this article, we briefly discuss the history of the computation of π and some formulas that mathematicians have discovered to represent this number. A detailed history of π can be found in [1] and [2]. Then, we describe the ideas and steps of Archimedes' derivation and close with some concluding remarks.

Keywords: pi, Archimedes, irrational number

A brief history of the number pi

The value of π was first calculated 4000 years ago. The ancient Babylonians and the Egyptians calculated an approximate value of π by actual physical measurements of the circumference or the area of a circle, and they estimated that π had a value close to 3.

About 1500 years later, the Greek mathematician Archimedes first used mathematics to estimate π and showed that its value lies between 22/7 and 223/71. The way he reasoned was as follows. He noted that a regular polygon circumscribed around a circle would have a perimeter *larger* than the circumference of the circle, while a regular polygon inscribed in the circle would have a *smaller* perimeter. He then observed that as one increased the number of sides of the polygon, the two perimeters close in on the circumference of the circle. Finally, he used Pythagoras's theorem to find the perimeters of the polygons and thus got upper and lower bounds for the value of π. Using a hexagon, a 12-sided polygon, a 24-sided polygon, a 48-sided polygon and then a 96-sided polygon, he proved that $223/71 < \pi < 22/7$. In this article, we discuss the basic ideas behind his derivation. (In fact, we provide an improved lower bound.) The details of his original method can be found in [1].

During the 5th century CE, the Indian mathematician Aryabhata calculated a value of π that was accurate up to 3 decimal digits. Zu Chongzhi, a Chinese mathematician and astronomer, calculated an approximate value of π using a 24576-gon, about 7 centuries after Archimedes. His estimated value, 355/113, is approximately equal to 3.14159292. The Greek letter π was first introduced by William Jones in 1706. It was derived from the first letter of the Greek word 'perimetros,' meaning circumference.

Formulas for pi

Attempting to find the value of π by actual physical measurement of the circumference and diameter of a circle would involve a lot of errors and not be particularly reliable. There are many different formulas that mathematicians have discovered for representing π. Here are some examples [1]–[5].

Infinite series representations.

(A) The Madhava-Gregory-Leibniz series (1300-1700 CE):

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

(B) Nilakantha series (1500 CE):

$$\frac{\pi}{4} = \frac{3}{4} + \frac{1}{2 \times 3 \times 4} - \frac{1}{4 \times 5 \times 6} + \frac{1}{6 \times 7 \times 8} - \frac{1}{8 \times 9 \times 10} + \cdots$$

Using this series, we can get accurate estimates for π by including more terms in the series.

Continued fractions representation. The following form was obtained by Brouncker (1660 CE):

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}$$

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Infinite product representation. The following product form was found by Wallis (1656 CE):

$$\frac{\pi}{2} = \left(\frac{2}{1}\cdot\frac{2}{3}\right) \times \left(\frac{4}{3}\cdot\frac{4}{5}\right) \times \left(\frac{6}{5}\cdot\frac{6}{7}\right) \times \left(\frac{8}{7}\cdot\frac{8}{9}\right) \times \cdots$$

Infinite nested radical representation. This involves taking an infinite number of nested square roots and was first found by Vieté (1593 CE):

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \times \frac{\sqrt{2+\sqrt{2}}}{2} \times \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \times \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2} \times \cdots$$

As we will see, Archimedes' method leads to a similar representation.

Archimedes's method for estimating pi

As discussed above, the basic idea is quite simple. We draw a regular polygon with n sides that circumscribes a circle with diameter 1. Let the perimeter of such a polygon be denoted by C_n. We also inscribe a regular polygon with n sides in the circle. Let its perimeter be denoted by c_n. Then, if C denotes the circumference of the circle, it is clear that we have

$$c_n < C < C_n. \tag{2}$$

As we increase n, both c_n and C_n should get closer to C. We illustrate this in Figure 1 for the cases with $n = 3$, 6 and 12. It is clear that the polygon with 12 sides (dodecagon) has a circumference that is the closest to the circle.

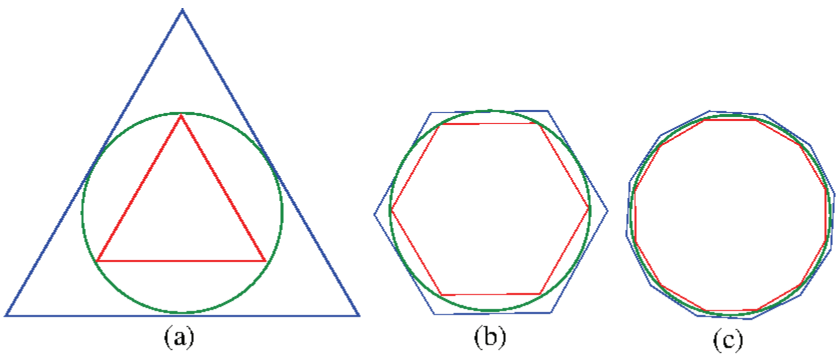

Figure 1. Here we show polygons of n sides inscribed (red) and circumscribed (blue) around a circle: (a) triangle ($n = 3$), (b) hexagon ($n = 6$), (c) dodecagon ($n = 12$)

Therefore, if we can find a way to compute the perimeter of an n-sided polygon, we can arrive at an accurate value of π simply by taking a large value for n. We now need a method to find the perimeter of an n-sided polygon. We can do this using trigonometry and the Pythagoras theorem though we must be careful not to use any result that already uses the value of π. From Figure 2a and Figure 2b, it is clear that the perimeters of the inscribed and circumscribed polygons for a circle of diameter 1 are respectively given by

$$c_n = n \cdot \sin\frac{180^\circ}{n}, \qquad C_n = n \cdot \tan\frac{180^\circ}{n}. \tag{3}$$

It turns out that for some choices of n, it is easy to evaluate c_n and C_n.

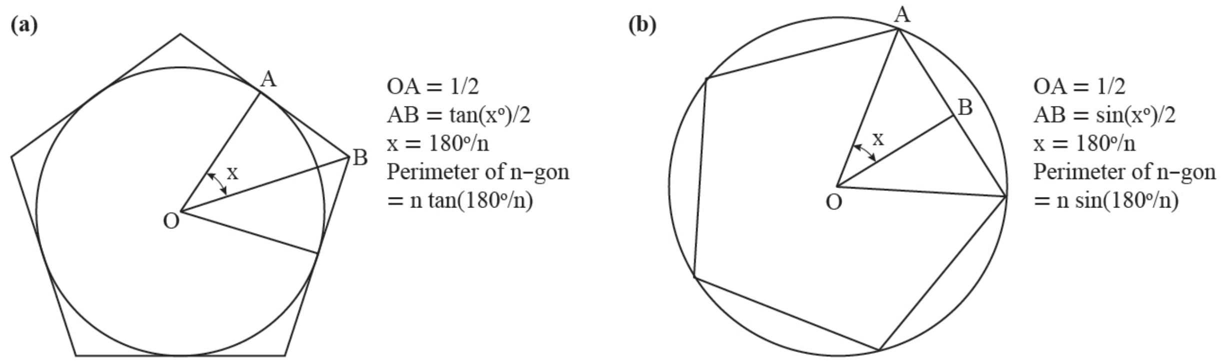

Figure 2. Here we show the calculation of (a) C_n and (b) c_n for $n = 5$

For example, take $n = 3$. Using Pythagoras theorem one can easily prove that $\sin 60° = \sqrt{3}/2$, so we get $c_3 = 3\sqrt{3}/2$. Next, using the formulas

$$\cos x = \sqrt{1 - \sin^2 x}, \quad \tan x = \frac{\sin x}{\cos x}, \tag{4}$$

we get $\cos 60° = 1/2$ and $\tan 60° = \sqrt{3}$. Hence, we get $C_3 = 3\sqrt{3}$.

If we now double the number of sides, we get $n = 6$, and we need the values of $\sin 30°$ and $\tan 30°$ to find the perimeters of the inscribed and circumscribed hexagons. Now $\cos 30°$ is easy to find from $\cos 60°$ using the trigonometric identity

$$\cos x = \sqrt{\frac{1 + \cos 2x}{2}}. \tag{5}$$

This then gives us $\cos 30° = \sqrt{3}/2$ and $\sin 30° = 1/2$. And then we go to $n = 12$ for which we use the same rule to find $\cos 15° = \sqrt{2 + \sqrt{3}}/2$ and $\sin 15° = \sqrt{2 - \sqrt{3}}/2$. So the idea is to keep doubling n and each time we can find the cosine, sine and tangent of the relevant angle by using the formulas in (4) and (5). The results for $n = 3, 6, 12, 24, 48, 96$ are given in Table 1. We see a pattern in the expressions and after some time we can guess the next entry in the table. We also plot the numerical values (up to the 8th decimal place) and see that they become closer to π.

In Figure 3a we show how c_n and C_n for a circle with unit diameter change as we make n larger. Figure 3b shows a zoomed in version of Figure 3a, the values $22/7 = 220/70$ and $223/71$ that were given by Archimedes, and also the true value of π. We see that $22/7$ is just a bit larger than C_{96} while $223/71$ is just a bit smaller than c_{96}.

It is clear that to get an accurate value of π, we have to take a very large value of n. In Figure 3 we see that even with a 96-gon the difference is in the third decimal place. To 12 decimal places it is known that $\pi = 3.141592653589$, while from our formula we find that $C_{24576} = 3.141592670702$ and $c_{24576} = 3.141592645034$. The Chinese mathematician Zu Chongzhi in fact used these results for the 24576-gon to suggest the approximate rational value $355/113$ which is about 3.141592920354.

Rational approximations

We see from Table 1 that the perimeter of the polygon typically involves taking multiple square roots and is an irrational number. An interesting question is to find a rational number that is very close to this irrational number. Consider the value $c_{96} \approx 3.14103195$. A rational number approximating this (called a

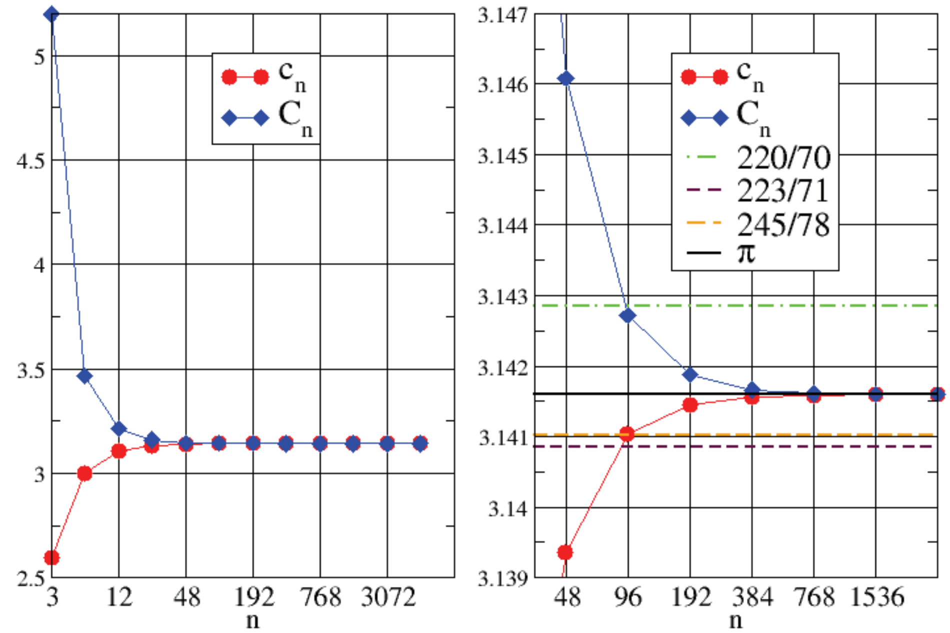

Figure 3. The values obtained for c_n and C_n plotted against n. The right panel shows a close-up so that we can see the values 22/7 and 223/71 obtained by Archimedes and also the true value of π. The improved lower bound 245/78 is also plotted.

n	$c_n = n \cdot \sin(180°/n)$	Approx value	$C_n = n \cdot \tan(180°/n)$	Approx value
3	$3\sqrt{3}/2$	2.59807621	$3\sqrt{3}$	5.19615242
6	3	3	$2\sqrt{3}$	3.46410161
12	$12\left(\frac{\sqrt{2-\sqrt{3}}}{2}\right)$	3.10582854	$12\left(\frac{\sqrt{2-\sqrt{3}}}{\sqrt{2+\sqrt{3}}}\right)$	3.21539031
24	$24\left(\frac{\sqrt{2-\sqrt{2+\sqrt{3}}}}{2}\right)$	3.13262861	$24\left(\frac{\sqrt{2-\sqrt{2+\sqrt{3}}}}{\sqrt{2+\sqrt{2+\sqrt{3}}}}\right)$	3.15965994
48	$48\left(\frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}}{2}\right)$	3.13935020	$48\left(\frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}}{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}\right)$	3.14608622
96	$96\left(\frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}{2}\right)$	3.14103195	$96\left(\frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}\right)$	3.14271460

Table 1.

rational approximant) is 314103/100000. But can we get a rational approximant with a smaller denominator? Archimedes obtained the value $223/71 \approx 3.140845$ which has a smaller denominator and is still a good approximation. We are not sure how Archimedes made this guess.

However, one approach to find a good rational approximation for any number is to write it in the form of a *continued fraction expansion* that we then truncate after some term. We discuss this idea briefly and apply it to our problem.

A continued fraction expansion is given by the following form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},$$

where $a_1, a_2, \ldots$ are positive integers. For an irrational number, the series goes on forever, while for a rational number it ends after a finite number of terms. If we specify a number by its numerical value (to some decimal places), there is a simple procedure to write down its continued fraction expansion. For illustration let us consider the number $3.14 = 157/50$. We see that $a_0 = 3$. The remaining part of the expansion has the value $7/50$. To find a_1, we have to take the integer part of its reciprocal, i.e., $50/7$. This gives us $a_1 = 7$. Repeating the procedure with the remainder, that is $1/7$, we find $a_2 = 7$. Hence we can write

$$3.14 = 3 + \cfrac{1}{7 + \cfrac{1}{7}}.$$

So we get the series of rational approximants for 3.14 as: 3/1, 22/7, 157/50.

If we perform this procedure for $c_{96} \approx 3.14103195$, we get

$$\frac{314103195}{100000000} = 3 + \cfrac{1}{7 + \cfrac{1}{11 + \cfrac{1}{25 + \cfrac{1}{1 + \cfrac{1}{25 + \cfrac{1}{1 + \cfrac{1}{27 + \cfrac{1}{13}}}}}}}}.$$

Truncating the above series at various orders, we get the rational approximants 3/1, 22/7, 245/78, 6147/1957, and so on. This does not include Archimedes' value of 223/71. However, notice that $245/78 \approx 3.141025641$ is in fact a better approximation to c_{96} than 223/71. Moreover we see that $245/78 < c_{96}$ and so we can use it as a lower bound. Applying the same procedure to C_{96} gives us the approximants 3/1, 22/7, 3149/1002, and so on. In this case we see that $22/7 > C_{96}$ and so we can use this value as an upper bound. Hence we finally get the somewhat improved bound

$$\frac{245}{78} < \pi < \frac{22}{7}, \quad \text{i.e.,} \quad 3\frac{11}{78} < \pi < 3\frac{1}{7}.$$

The lower bound here is better than that given by Archimedes, in the sense that

$$\frac{223}{71} < \frac{245}{78} < \pi < \frac{22}{7}, \quad \text{i.e.,} \quad 3\frac{10}{71} < 3\frac{11}{78} < \pi < 3\frac{1}{7}.$$

If we find the continued fraction series for $C_{24576} = 3.14159267$, we get the approximants $3/1$, $22/7$, $333/106$, $355/113$, . . . The last number is the approximation discovered by Zu Chongzhi.

Conclusion

In this article, we have described Archimedes's method of determining the value of π by approximating the circumference of a circle of unit diameter by the perimeters of inscribed and circumscribed regular polygons. As we increase the number of sides, we get more accurate approximations for π. As we have shown, it turns out to be easy to find the required perimeters if we restrict ourselves to polygons with number of sides $n = 3 \times 2^k$, with $k = 0, 1, 2, 3, \ldots$, i.e., $n = 3, 6, 12, 24, 48, \ldots$. Archimedes went up to $n = 96$ and from this deduced the approximate value $22/7$ that is widely used today. Zu Chongzhi went up to $n = 24576$ and obtained the value $355/113$ which is a better approximation of π. Today we can use a calculator or computer to find the precise numerical value of the perimeter of a polygon, which typically involves finding a lot of square roots. It must have been very difficult in the times of Archimedes and Chongzhi to compute these numbers by hand. It would have required a lot of clever thinking to arrive at simple rational approximations such as $22/7$ and $355/113$, and it is an interesting question as to how exactly they arrived at these results [1].

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