Fun with Equivalent Fractions

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Start with any single digit pair of equivalent fractions such as $\frac{1}{2}$ and $\frac{2}{4}$ or $\frac{2}{3}$ and $\frac{4}{6}$. In general, let us write this as $\frac{a}{b} = \frac{c}{d}$. Note that we can also include special cases such as $\frac{a}{b} = \frac{b}{c}$.

Using the properties of proportions ('componendo' and 'dividendo'), we can write $\frac{a}{b} = \frac{c}{d}$ as

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

Multiplying both sides by $\frac{11}{9}$, we get:

$$\frac{11a+11b}{9a-9b} = \frac{11c+11d}{9c-9d}.$$

Applying componendo-dividendo again we get,

$$\frac{20a+2b}{2a+20b} = \frac{20c+2d}{2c+20d}.$$

Removing the common factor 2, we get

$$\frac{10a+b}{a+10b} = \frac{10c+d}{c+10d}.$$

This may be written as a product in the following form:

$$(10a+b)(10d+c) = (10b+a)(10c+d).$$

Now observe that $10a+b$, $10b+a$, $10c+d$ and $10d+c$ are actually the 2-digit numbers $\overline{ab}$, $\overline{ba}$, $\overline{cd}$ and $\overline{dc}$.

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Therefore, by replacing a, b, c, d by suitable digits from different pairs of equivalent fractions $\frac{a}{b} = \frac{c}{d}$, we get many interesting equalities. For example, from the equivalent fractions $\frac{2}{3} = \frac{4}{6}$, we get

$$23 \times 64 = 32 \times 46.$$

Thus we have a case where the product of two 2-digit numbers is the same as the product of the numbers obtained by reversing the order of the digits.

We can thus generate the list of all such pairs, using nothing more than the properties of proportions. It turns out that there are 14 such pairs of numbers as listed below:

$$\begin{aligned}
12 \times 42 &= 21 \times 24 = 504 \\
12 \times 63 &= 21 \times 36 = 756 \\
12 \times 84 &= 21 \times 48 = 1008 \\
24 \times 63 &= 42 \times 36 = 1512 \\
24 \times 84 &= 42 \times 48 = 2016 \\
36 \times 84 &= 63 \times 48 = 3024 \\
13 \times 62 &= 31 \times 26 = 806 \\
13 \times 93 &= 31 \times 39 = 1209 \\
26 \times 93 &= 62 \times 39 = 2418 \\
14 \times 82 &= 41 \times 28 = 1148 \\
23 \times 64 &= 32 \times 46 = 1472 \\
23 \times 96 &= 32 \times 69 = 2208 \\
46 \times 96 &= 64 \times 69 = 4416 \\
34 \times 86 &= 43 \times 68 = 2924
\end{aligned}$$

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