Problem Corner

Addendum to "Six Problems on Area and Perimeter"

SHAILESH SHIRALI

We present here geometric solutions to the first two questions in "Six problems on area and perimeter". The solution for the second question involves a beautiful 'proof without words' (PWW).

Problem 1. Figure 1 (a) shows a square $ABCD$ inscribed in a circle with centre O. P is the midpoint of minor arc AB. Find the ratio of the areas of $\triangle APB$ and $\triangle AOB$.

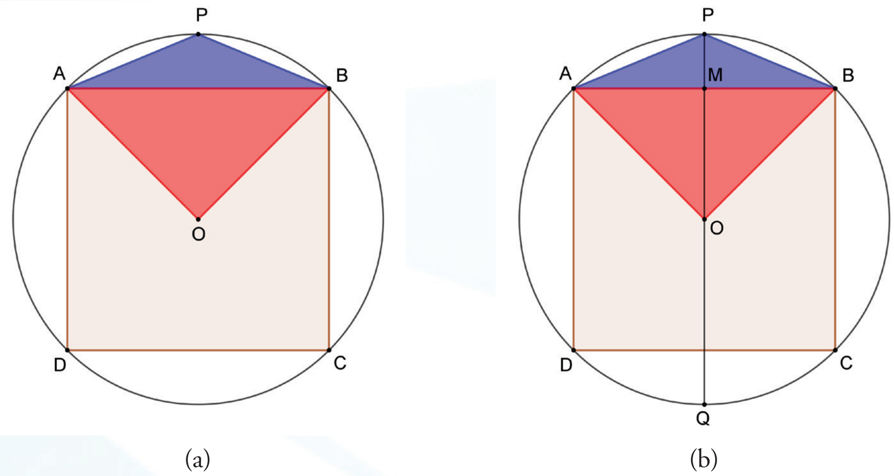


Figure 1.

Keywords: Intersecting chords theorem

Solutions. We take the circle to have radius 1. Let PQ be the diameter through P; see Figure 1 (b). The midpoint M of AB lies on PO. Since $\triangle AMO$ is isosceles and right-angled, it follows (using Pythagoras's theorem) that $OM = 1/\sqrt{2}$ and therefore $PM = 1 - 1/\sqrt{2}$. It follows that

$$\frac{PM}{OM} = \frac{1 - 1/\sqrt{2}}{1/\sqrt{2}} = \sqrt{2} - 1.$$

Hence area of $\triangle APB$: area of $\triangle AOB = \sqrt{2} - 1 : 1$. (Areas of triangles on the same base are in the same ratio as their heights.)

Alternative solutions. Another approach is to use the intersecting chords theorem. This is the statement that if two chords PQ and RS in a circle intersect at point M, then $PM \cdot MQ = RM \cdot MS$. This may be shown using the fact that $\triangle PMR$ and $\triangle SMQ$ are similar to each other.

We refer again to Figure 1 (b). Let x be the length of PM. To find x, note that $AM = 1/\sqrt{2}$. Also, $MQ = 2 - x$. The intersecting chords theorem tells us that

$$\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = x \cdot (2 - x), \quad \therefore\ 2x^2 - 4x + 1 = 0, \quad \therefore\ x \in \left\{ \frac{\sqrt{2}-1}{\sqrt{2}},\ \frac{\sqrt{2}+1}{\sqrt{2}} \right\}.$$

The second solution does not fit here, as we must have $x < 1$. Hence

$$x = \frac{\sqrt{2}-1}{\sqrt{2}}, \quad \therefore\ 1 - x = \frac{1}{\sqrt{2}}.$$

Therefore

$$\frac{x}{1-x} = \frac{\sqrt{2}-1}{1}.$$

Hence area of $\triangle APB$: area of $\triangle AOB = \sqrt{2} - 1 : 1$, as earlier.

Problem 2. Figure 2 (a) shows a square $ABCD$ inscribed in a circle with centre O. Points P, Q are situated on the minor arc AB such that $AP = PQ = QB$. Find the ratio of the areas of $\triangle AOB$ and quadrilateral $APQB$.

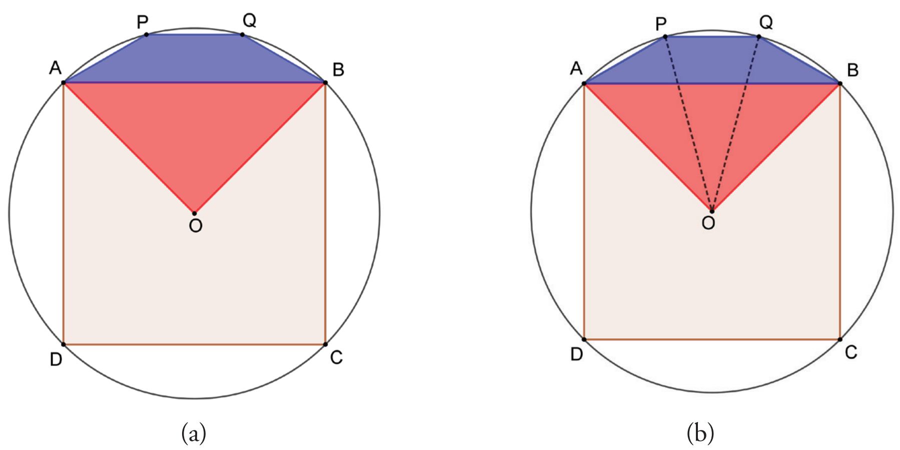


Figure 2.

Solutions. Join OP and OQ, and note that $\measuredangle AOP = \measuredangle POQ = \measuredangle QOB = 30°$. See Figure 2 (b). Four copies of pentagon $OAPQB$ can be used to make up a regular 12-sided polygon (i.e., a dodecagon) inscribed in the same circle, as shown in Figure 3. If we can find the area of this polygon, we will know the area of pentagon $OAPQB$, and from this we can find the desired ratio, as the area of $\triangle AOB$ is easily computed.

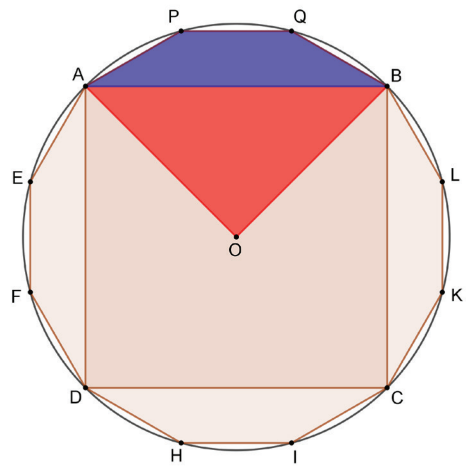


Figure 3. *AEFDHICKLBQP* is a regular 12-sided polygon (a dodecagon)

Before proceeding, we make an important observation about triangle APQ: it is isosceles, with angles of $15°, 15°, 150°$. Two copies of this triangle can be joined along their longest side to make a rhombus (a 'diamond') with angles of $30°, 150°$. See Figure 4.

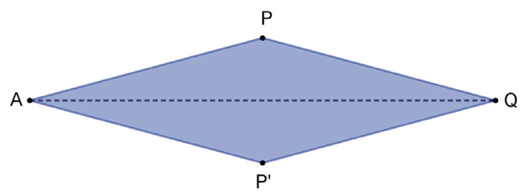


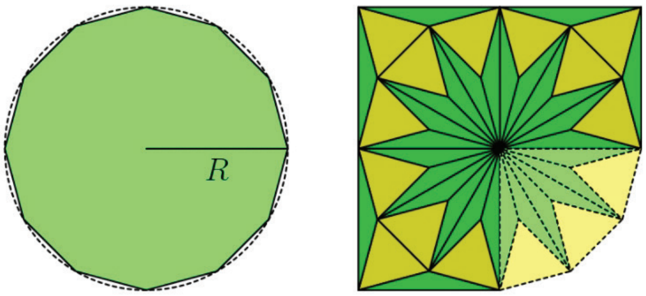
Figure 4.

Now there is a beautiful 'proof without words' which shows that the area of a regular 12-sided polygon inscribed in a circle of radius R is $3R^2$. See Figure 5. Observe how 24 congruent copies of $\triangle APQ$ and 12 congruent copies of the equilateral triangle with side AP fit together beautifully to make three-quarters of a square whose side is the diameter of the circumscribing circle. Note that a few extra copies of the two kinds of triangles are needed to make the full square, as shown in the right-hand corner (using a lighter colour; 6 congruent copies of $\triangle APQ$ and 3 congruent copies of the equilateral triangle with side AP). Since the area of the full square is $(2R)^2 = 4R^2$, it follows that the area of the 12-sided polygon is $3/4$ of this, i.e., $3R^2$.

Using this result, and referring once again to Figure 2 (b), we see that the area of $OAPQB$ is

$$\frac{1}{4} \cdot 3 \cdot 1^2 = \frac{3}{4}.$$

Since the area of $\triangle AOB$ is $1/2$, the area of quadrilateral $APQB$ is $3/4 - 1/2 = 1/4$. We deduce that area of quadrilateral $APQB$: area of $\triangle AOB = 1 : 2$.



The area of a dodecagon is $3R^2$, where R is the circumradius.

Figure 5. Source: https://artofproblemsolving.com/wiki/index.php?title=Proofs_without_words

Comment. Obviously, the solution for the second problem given in the article "Six problems on area and perimeter", using trigonometry, is far more compact and simple than the solution presented here! (This only goes to underscore the great power of trigonometry—something which is not always appreciated by us.) However, we have shared these alternative solutions here only to show how different ideas in mathematics can fit together to make a beautiful and composite whole. We hope you have enjoyed reading through these solutions.

SHAILESH SHIRALI is Director of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.