Problem Corner

On a method for Solving Cubic Equations

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In this short note, we discuss a method to solve cubic equations. It is based on a method of factorisation developed by Abdul Halim Sk., a school teacher of West Bengal, so we call it 'Halim's method of factorisation.'

It involves a certain degree of hit-&-trial (or 'guesswork'), and may be applied to cubic polynomials of the forms $x^3 + bx^2 + c$ and $x^3 + bx + c$. Here, b and c are integers.

Let us see how the approach works for the polynomial $x^3 + bx + c$. Suppose that

$$x^3 + bx + c = (x + p)(x^2 + qx + r).$$

The expression on the right side is equal to $x^3 + (q + p)x^2 + (r + pq)x + pr$. As this is identically equal to $x^3 + bx + c$, we may equate coefficients of like powers of x on both sides. We get:

$$\begin{aligned} q + p &= 0, \\ r + pq &= b, \\ pr &= c. \end{aligned}$$

These equalities yield $q = -p$ and $b = r - p^2$. Therefore we can rewrite the given polynomial as

$$x^3 + bx + c = x^3 + (r - p^2)x + pr.$$

Now we apply the above to solve a cubic equation.

Keywords: Cubic equation, polynomial, factorisation

The general form of a cubic equation is $ax^3 + 3bx^2 + 3cx + d = 0$.

We first reduce it to the standard form by the transformation $y = ax + b$. This removes the quadratic term, and we are left with the equation $y^3 + 3Hy + G = 0$ (for some H, G).

We factorize this using Halim's method:

$$y^3 + 3Hy + G = (y + p)(y^2 - py + r),$$

$$\text{where } 3H = r - p^2 \text{ and } G = pr.$$

For this, we must look for a pair of numbers p, r such that $3H = r - p^2$ and $G = pr$. This involves a certain amount of trial and error.

If we are easily able to find p and r, then by solving the linear equation $y + p = 0$ and the quadratic equation $y^2 - py + r = 0$, we find all three roots of $y^3 + 3Hy + G = 0$:

$$-p, \quad \frac{p \pm \sqrt{p^2 - 4r}}{2}.$$

Finally, from the relation $y = ax + b$, we get all the roots of the equation $ax^3 + 3bx^2 + 3cx + d = 0$.

We demonstrate this using two examples.

Example 1

Take the equation $x^3 - 6x - 9 = 0$. We must look for a pair of numbers p, r such that $-6 = r - p^2$ and $-9 = pr$. By inspection we find $r = 3$ and $p = -3$, because $-9 = (-3) \times 3$ and $-6 = 3 - 3^2$. So:

$$\begin{aligned} & x^3 - 6x - 9 = 0 \\ \implies & x^3 - (3^2 - 3)x - 9 = 0 \\ \implies & (x - 3)(x^2 + 3x + 3) = 0. \end{aligned}$$

From $x - 3 = 0$ we get $x = 3$, and from $x^2 + 3x + 3 = 0$ we get $x = \frac{1}{2}(-3 \pm i\sqrt{3})$.

So the roots of the given equation are

$$\left\{3, \ \frac{-3 \pm i\sqrt{3}}{2}\right\}.$$

Example 2

Take the equation $x^3 - 12x + 65 = 0$. We must look for a pair of numbers p, r such that $-12 = r - p^2$ and $65 = pr$. By inspection we find $r = 13$ and $p = 5$, because $65 = 5 \times 13$ and $12 = 5^2 - 13$. So:

$$\begin{aligned} & x^3 - 12x + 65 = 0 \\ \implies & x^3 - (5^2 - 13)x + 65 = 0 \\ \implies & (x + 5)(x^2 - 5x + 13) = 0. \end{aligned}$$

From $x + 5 = 0$ we get $x = -5$, and from $x^2 - 5x + 13 = 0$ we get $x = \frac{1}{2}(5 \pm 3i\sqrt{3})$.

So the roots of the given equation are

$$\left\{-5, \ \frac{5 \pm 3i\sqrt{3}}{2}\right\}.$$

Closing remarks

Will this method always work? Given the cubic equation $y^3 + 3Hy + G$, it should be clear that the success of this approach depends on our easily finding a pair of numbers p and r such that $3H = r - p^2$ and $G = pr$.

As noted above, this requires trial and error. Unfortunately, there is no straightforward way to find such a pair of numbers. If we try to do it systematically, by setting it up as an equation, we end up with the very equation that we had started with.

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