Triangles with One Angle Twice Another

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Triangles with one angle equal to another are familiar objects; a lot is known about them. What can be said about triangles in which one angle is twice another? Can such triangles be characterised in any other way? We explore these questions in this article.

We have the following striking and compact result about triangles in which one angle is twice another (see Figure 1).

Theorem 1. In any $\triangle ABC$, the following is true:

$$\measuredangle A = 2\measuredangle B \iff a^2 = b(b+c). \tag{1}$$

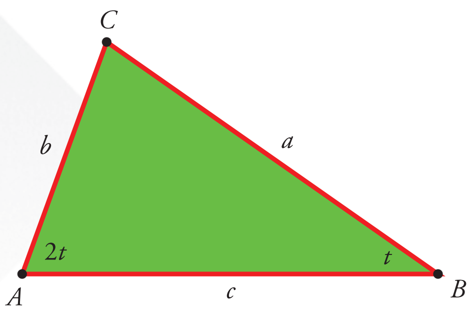

Figure 1. Triangle ABC with $\measuredangle A = 2\measuredangle B$

Observe that the result is an "if and only if" statement. We consider the forward and reverse implications separately.

We offer two different kinds of proofs of the result. It is interesting that the reverse implication presents a greater challenge using either approach.

Proof using trigonometry. We make use of the sine rule for triangles and the fact that supplementary angles have equal sine values.

Forward implication. We must prove that if $\measuredangle A = 2\measuredangle B$, then $a^2 = b(b+c)$. Let $\measuredangle B = t$; then $\measuredangle A = 2t$ and $\measuredangle C = 180^\circ - 3t$. We therefore have:

$$\frac{a}{\sin 2t} = \frac{b}{\sin t} = \frac{c}{\sin 3t}. \tag{2}$$

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Multiplying through by $\sin t$ and remembering the double angle and triple angle identities, we get:

$$\frac{a}{2\cos t} = b = \frac{c}{3 - 4\sin^2 t}.$$

Since $3 - 4\sin^2 t = 4\cos^2 t - 1$, we have:

$$2\cos t = \frac{a}{b}, \qquad 4\cos^2 t - 1 = \frac{c}{b}. \tag{3}$$

We may easily eliminate t from the above two equalities:

$$\frac{c}{b} = \frac{a^2}{b^2} - 1, \qquad \therefore \; bc = a^2 - b^2,$$

and so

$$a^2 = b(b+c). \tag{4}$$

Reverse implication. We must prove that if $a^2 = b(b+c)$, then $\measuredangle A = 2\measuredangle B$. Invoking the sine rule again, the given equality leads to

$$\begin{aligned} \sin^2 A &= \sin B \cdot (\sin B + \sin C), \\ \therefore \; \sin^2 A - \sin^2 B &= \sin B \cdot \sin C. \end{aligned} \tag{5}$$

We now invoke the following striking and beautiful trigonometric identity (which looks extremely surprising at first glance, as it looks just like the "difference of two squares" identity):

$$\sin^2 A - \sin^2 B = \sin(A+B) \cdot \sin(A-B).$$

We also have $\sin(A+B) = \sin C$. Hence from (5) we get $\sin B \cdot \sin C = \sin C \cdot \sin(A-B)$, and so (since $\sin C \neq 0$),

$$\sin B = \sin(A-B). \tag{6}$$

From (6) the following two possibilities arise:

- the angles B and $A - B$ are equal; OR
- the angles B and $A - B$ are supplementary.

The second possibility leads to $A = 180^\circ$, which is absurd. Therefore we must have $B = A - B$, which leads to $A = 2B$, as required. □

Proof using 'pure' geometry. Examining the form of $a^2 = b(b+c)$, we are led to expect that the proof will involve working with suitably constructed similar triangles. This is because the expression $a^2 = b(b+c)$ may be written as $a/b = (b+c)/a$, and this immediately suggests looking for a pair of similar triangles. (Actually, there is another possible line of inquiry, but we will say something about this at the end.)

Forward implication. We must prove that if $\measuredangle A = 2\measuredangle B$, then $a^2 = b(b+c)$. Since $A = 2B$, it makes sense to draw the angle bisector of $\measuredangle BAC$, as this will give us an angle equal to $\measuredangle CBA$. Drawing this angle bisector, we obtain Figure 2. The angle bisector meets side CB at D. Let $CD = x$; then $DB = a - x$.

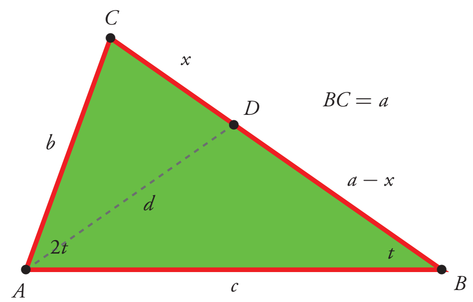

Figure 2. Triangle ABC with $\measuredangle A = 2\measuredangle B$

In Figure 2, $\triangle DAB$ is isosceles, so $DA = DB$, i.e., $d = a - x$. Also, $\measuredangle CDA = 2t$ (note that it is an exterior angle to $\triangle DAB$). This means that $\triangle CAD$ has the same set of angles as $\triangle CBA$ (namely: $180^\circ - 3t$, t, $2t$). The two triangles are therefore similar to each other, so their sides (namely: $\{x, b, d\}$ and $\{b, a, c\}$, respectively) must be in proportion. That is:

$$\frac{x}{b} = \frac{b}{a} = \frac{a-x}{c}. \tag{7}$$

These equalities give:

$$x = \frac{b^2}{a}, \qquad \therefore\ \frac{b}{a} = \frac{a - b^2/a}{c} = \frac{a^2 - b^2}{ac},$$

and therefore

$$b = \frac{a^2 - b^2}{c}, \qquad \therefore\ a^2 = b^2 + bc = b(b+c). \tag{8}$$

Reverse implication. This presents a greater challenge. Starting with the expression $a^2 = b(b+c)$, we must construct a pair of similar triangles. The challenge here is to make geometric sense of the expression $b + c$, which is a sum of two lengths which do not even lie in a straight line. We shall solve the problem using a suitable construction (Figure 3).

Extend side CA of $\triangle ABC$ to E such that $AE = AB$, i.e., $AE = c$. This results in a segment CE which has length $b + c$. Join BE. Now write the given relation $a^2 = b(b+c)$ as

$$\frac{a}{b} = \frac{b+c}{a}.$$

With reference to Figure 3, this states that

$$\frac{CB}{CA} = \frac{CE}{CB}. \tag{9}$$

Relation (9) immediately tells us that

$$\triangle CBA \sim \triangle CEB, \tag{10}$$

(note the order in which we have labelled the vertices—it indicates the vertex correspondence) and hence that we must have the following angle equalities:

$$\measuredangle CBA = \measuredangle CEB, \tag{11}$$
$$\measuredangle CAB = \measuredangle CBE. \tag{12}$$

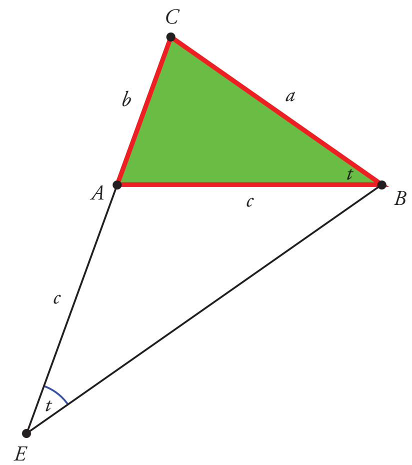

Figure 3. Triangle ABC with $a^2 = b(b+c)$

Relation (11) tells us that $\measuredangle ABE = t$ (since $AE = c = AB$), and therefore that $\measuredangle CBE = 2t$.

Relation (12) combined with the above finding tells us that $\measuredangle CAB = 2t$.

Hence $\measuredangle CAB = 2\measuredangle CBA$, i.e., $\measuredangle A = 2\measuredangle B$. □

A pure geometry solution using a single figure. Examining the above two proofs (for the forward and reverse implications), the reader will notice that we have used different constructions for the two proofs. This may seem unsatisfactory. Is it possible to use a single figure to prove both the implications? We successfully answer this challenge. (It turns out to be simpler than expected!)

Figure 4 shows $\triangle ABC$ in which we extend side CA beyond A to E so that $AE = AB$. We then draw $\triangle ABE$. Since $AE = AB$, we have $\measuredangle ABE = \measuredangle AEB$. We now consider the claim. The reverse implication has already been dealt with above, so we consider only the forward implication.

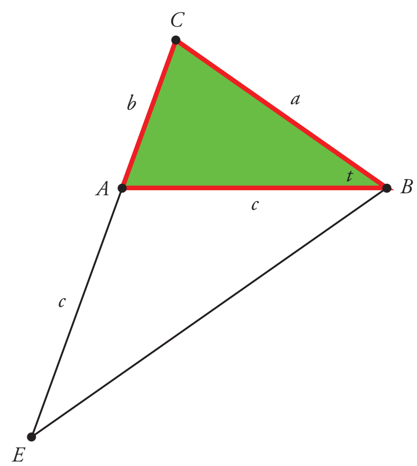

Figure 4. Triangle ABC with CA extended to E so that $AE = AB$

Suppose that $\measuredangle A = 2\measuredangle B$, i.e., $\measuredangle A = 2t$. Since $\measuredangle CAB = \measuredangle ABE + \measuredangle AEB = 2\measuredangle AEB$, it follows that $\measuredangle AEB = t = \measuredangle ABE$.

Observing that the angles of $\triangle CEB$ are identical to those of $\triangle CBA$ (namely: $180^\circ - 3t$, t, $2t$), we see that the two triangles are similar to each other. Therefore their sides are in proportion. From this it follows that

$$\frac{a}{b} = \frac{b+c}{a}. \tag{13}$$

Simplifying (13), we obtain $a^2 = b(b+c)$, as required. $\square$

Postscript: A trigonometric analysis that yields a strange conclusion. In the same spirit as the geometric analysis presented above, can we do a trigonometric analysis that yields both the forward and reverse implications in a single movement? This too is possible, but there is a slight twist. The 'twist' occurs with the isosceles case.

Consider first the case when the triangle has $\measuredangle B = \measuredangle C$ (i.e., $b = c$) and $\measuredangle A = 2\measuredangle B$. That is, we have $\measuredangle A = 2\measuredangle B = 2\measuredangle C$. The triangle is now isosceles right-angled, with angles $90^\circ, 45^\circ, 45^\circ$. So we have $a/b = \sqrt{2} = a/c$, and the relation $a^2 = b(b+c)$ holds.

Conversely, if we have $b = c$ together with $a^2 = b(b+c)$, then $a^2 = 2b^2$, so $a/b = \sqrt{2} = a/c$, leading to the triangle having angles $90^\circ, 45^\circ, 45^\circ$, in which case the relation $\measuredangle A = 2\measuredangle B$ holds.

Therefore, the case with $b = c$ satisfies the conditions of the theorem and need not be considered further. In the analysis below, we explicitly exclude the case when $\measuredangle B = \measuredangle C$. We have:

$$\begin{aligned}
\measuredangle A = 2\measuredangle B &\iff \sin A = \sin 2B \qquad \text{(because we have } \measuredangle B \neq \measuredangle C\text{)} \\
&\iff \sin A = 2\sin B \cdot \cos B \\
&\iff \frac{a}{b} = 2\cos B \qquad \text{(by invoking the sine rule)} \\
&\iff \frac{a}{b} = \frac{a^2 + c^2 - b^2}{ac} \qquad \text{(by invoking the cosine rule)} \\
&\iff a^2 c - b\left(a^2 + c^2 - b^2\right) = 0 \\
&\iff a^2(c-b) - b\left(c^2 - b^2\right) = 0 \\
&\iff (c-b) \cdot \left(a^2 - b(b+c)\right) = 0 \\
&\iff a^2 - b(b+c) = 0 \qquad \text{(because we have } b \neq c\text{)}.
\end{aligned}$$

Therefore we have $\measuredangle A = 2\measuredangle B \iff a^2 = b(b+c)$, and the theorem is proved. $\square$

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