Student Corner

Factorising Non-monic Quadratic Equations

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In the November 2014 issue of At Right Angles, author Shashidhar Jagadeeshan, in the article "Completing the Square . . . A powerful technique, not a feared enemy!" talked about completing the square, quadratics having the shape of a parabola . . In this article, student Anushka Tonapi explains a few methods of solving non-monic quadratic equations.

A **polynomial** in a variable x has the form $ax^n + bx^{n-1} + \cdots + k$ (for some constants $a, b, \ldots, k$ and some positive integer n). A quadratic expression has the form $ax^2 + bx + c$, where a, b, c are constants, with $a \neq 0$. If $a = 1$, the quadratic is **monic**, and if $a \neq 1$, it is **non-monic**. When we equate this to 0, it becomes a **quadratic equation**. Solving a quadratic equation means finding its **roots**. Some widely-used techniques to find the roots of quadratic equations are:

1. The Quadratic Formula
2. Completing the Square
3. Factorisation

There are other interesting methods, each with its advantages and disadvantages:

1. Lyszkowski's / Modified Trial and Error Method
2. Vedic Mathematics Method
3. Slide and Divide Method
4. Monic / Scaling Method
5. Po-Shen Loh's Method

We illustrate these methods for the quadratic expression $ax^2 + bx + c$ (where a, b, c are given integers).

Keywords: Quadratics, Factorisation, Multiple methods of solution

Lyszkowski's / Modified Trial and Error Method

We begin by finding constants y and z such that $yz = ac$ and $y + z = b$. Next, we write the form $\frac{(ax+y)(ax+z)}{a}$. Finally, we take a common factor p and q out from each bracket in the numerator, where $pq = a$, getting $\frac{p(qx+y)\,q(px+z)}{a}$. The result: $(qx+y)(px+z) = ax^2 + bx + c$.

Example: Factorise $6x^2 - 5x - 4$.

Solution: We find that $(y, z) = (-8, 3)$. We get $\frac{(6x-8)(6x+3)}{6}$, and we can factorise this further by taking 2 and 3 out of the numerator: $\frac{2(3x-4)\,3(2x+1)}{6}$, which simplifies to $(3x-4)(2x+1)$.

Vedic Mathematics Method

We start by finding numbers p, q that add to b and multiply to ac. The constants p, q must form the proportion $a : p = q : c$. Then, we simplify any one of the ratios by reducing it to lowest terms. This ratio provides the linear coefficient and constant for the first factor. The process is called *Anurupyena Sutra* (अनुरूप्येण सूत्र).

We obtain the linear coefficient of the second factor by dividing a by that of the first factor and get the constant of the second factor by dividing c by the constant in the first factor. The process is known as *Adyamadyenantyamantya* (आद्यमाद्ये नान्त्यमन्त्येन).

The Vedic Mathematics[1] method results in a clever way of indirectly factorising the quadratic by observing the ratios between its coefficients.

Example: Factorise $15x^2 + 26x + 8$.

Solution: We look for two constants that multiply to give $ac = 120$ and add to give $b = 26$; we find that they are 20 and 6. Setting up our ratio, we get $\frac{15}{20} = \frac{6}{8}$. Simplifying, we get $\frac{3}{4} = \frac{6}{8}$.

Anurupyena gives us the first factor: $3x + 4$. Next, we perform *Aadyamaadyenaantyamatyena*. Dividing 15 by 3 gives 5, and dividing 8 by 4 gives 2; so we obtain the second factor: $5x + 2$. Therefore, $15x^2 + 26x + 8$ can be factorised as $(3x+4)(5x+2)$ using the Vedic Method.

Slide and Divide Method

We 'slide' a to the right and multiply by c to get $x^2 + bx + ac$, a monic quadratic. We factor this in the usual manner – factoring monics is easier than factoring non-monics. Suppose this can be factored as $(x+p)(x+q)$. Next, we divide each of the constants in the factorisation by a:

$$\left(x + \frac{p}{a}\right)\left(x + \frac{q}{a}\right).$$

Multiplying suitably, we get the required answer.

1 Vedic Mathematics is a way of performing operations based on some aphorisms constructed by Swami Bharati Krishna Tirtha (a saint of Shankaracharya order) in the early 20th century CE. It must not be confused with the mathematics contained in the Vedic texts written thousands of years ago.

Example: Factorise $4x^2 + 17x + 15$.

Solution: We 'slide' 4 to the right and get $x^2 + 17x + 60$. Factoring, we get $(x+5)(x+12)$. Now, to compensate for our earlier step, we divide each constant by 4; we get:

$$\left(x+\frac{5}{4}\right)\left(x+\frac{12}{4}\right) = \left(x+\frac{5}{4}\right)(x+3) = \frac{1}{4}(4x+5)(x+3).$$

From this we get the required answer $(4x+5)(x+3)$.

Monic/Scaling Method

We start by equating $ax^2 + bx + c$ to a dummy variable y to keep track of all our operations. Next, we multiply both sides by a to make the first term on the right-hand side a perfect square:

$$ay = a^2x^2 + abx + ac = (ax)^2 + b(ax) + ac.$$

Replacing ax by z, we get the monic quadratic

$$ay = z^2 + bz + ac.$$

Suppose that $z^2 + bz + ac$ can be factorised as $(z+p)(z+q)$. We have then:

$$ay = (z+p)(z+q) = (ax+p)(ax+q) = a(mx+n)(rx+s),$$

where $mr = a$, $an = p$, $as = q$. Dividing both sides by a, we get the factorisation

$$y = (mx+n)(rx+s).$$

Example: Factorise $9x^2 + 31x + 12$.

Solution: We set $9x^2 + 31x + 12$ equal to y and multiply both sides by 9:

$$9y = 81x^2 + 31(9x) + 12(9).$$

Replacing $9x$ by z, we rewrite this as $9y = z^2 + 31z + 108$. Factorising this monic quadratic the usual way, we get $(z+4)(z+27)$.

We have $9y = (z+4)(z+27) = (9x+4)(9x+27) = (9x+4)\,9\,(x+3)$. Dividing by 9, we get $y = (9x+4)(x+3)$. Therefore, $9x^2 + 31x + 12$ can be factorised as $(9x+4)(x+3)$.

Po-Shen Loh's Method

In 2019, Prof. Po-Shen Loh of Carnegie Mellon University came upon a method of finding the roots of a quadratic equation while thinking of an approach to introduce quadratic equations in his Expii Daily Challenge videos. To start with, he looks at the monic quadratic $x^2 + bx + c$. The method begins by reasoning that **if** one can find two constants r and s with sum $-b$ and product c, then we obtain the factorisation $x^2 + bx + c = (x-r)(x-s)$.

Now if two constants have sum $-b$, then their average is $-\frac{b}{2}$, so the two numbers must be of the form $-\frac{b}{2} \pm u$ for some constant u. We must find this constant. Multiplying the two numbers, we get $\left(-\frac{b}{2}+u\right)\left(-\frac{b}{2}-u\right) = \frac{b^2}{4} - u^2$. As this must be equal to c, we must have $\frac{b^2}{4} - c = u^2$. Solving for u

(we obtain two values, each the negative of the other), we obtain the two roots. It doesn't matter whether we use the positive or negative value of u; we get the same roots either way.

This results in a clean and straightforward process to factorise quadratics — with *no guessing*.

For the non-monic quadratic $ax^2 + bx + c$ where $a \neq 0$, we divide both sides by a, thus forming the monic quadratic $x^2 + \frac{b}{a}x + \frac{c}{a}$, and apply this method.

Example: Factorise $2x^2 - 4x - 5$.

Solution: Dividing by 2, we get $x^2 - 2x - \frac{5}{2}$.

We need two numbers with sum 2 and product $-\frac{5}{2}$. Let them be $1 + u$ and $1 - u$. Their product is $-\frac{5}{2}$. Therefore, $(1 + u)(1 - u) = -\frac{5}{2}$, which gives $1 - u^2 = -\frac{5}{2}$, and $u^2 = \frac{7}{2}$, or $u = \pm\sqrt{\frac{7}{2}}$. So the two roots are $1 \pm \sqrt{\frac{7}{2}}$.

Conditions in which these methods work and don't work

Of the methods described above, the first four (Lyszkowski's Modified Trial and Error Method; Vedic Mathematics Method; Slide and Divide Method; Monic / Scaling Method) work under the same conditions: they give the factors when the quadratic expression has real roots and can be factored using rational numbers. The fifth method (Po-Shen Loh's Method) can be used under all conditions, without any limitations.

References

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