Student Corner

Ellipses Hidden in Projectile Motion

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Introduction and Problem Statement

We study here the following very interesting problem associated with projectile motion.

Problem. A man standing at the origin of the Cartesian plane throws a projectile with a fixed velocity v at varying angles of projection θ. What figure does the point at which the projectile achieves maximum height (above the x-axis) trace as θ moves from 0 to π?

In other words, what is the shape formed when we mark the point where the projectile is highest from the ground, for all angles of projection from 0° to 180°? (Keep in mind that the quantity v is fixed. For the sake of simplicity, we do not consider air resistance.)

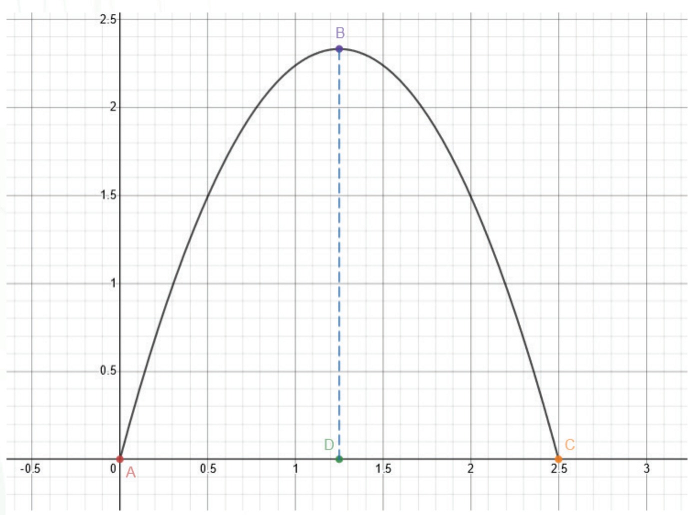

Figure 1. What figure does B trace as the trajectory of the projectile moves with v and g constant?

Keywords: Projectile motion, ellipse, Cartesian plane, parameter

I encountered this problem in an article by my favourite math blogger, Ben Orlin, who has a most humourous blog, *Math with Bad Drawings*. The link to his blog as well as the original tweet are given in the references.

(All graphs in the following article were made using the *Desmos* graphing calculator. The link to the graphs is given in the references.)

Parametrization and Required Formulae

So, how do we solve this problem? We first *parametrize* both the x-coordinate and y-coordinate of the moving point and then *eliminate* the changing variable θ to obtain an implicit equation in x and y which will describe this shape completely.

Parametrization means that we express how a point moves in space by expressing both its ordinate and abscissa in terms of some third variable (in this case θ). This means that the coordinates (x, y) of such a moving point satisfy

$$x = f(\theta), \quad y = g(\theta)$$

for functions f and g simultaneously. For simple examples, consider the following parametrizations.

Example 1.

$$x = 5k + 1, \quad y = 6k + 4.$$

This illustrates the simplest type of parametrization. Solving this is trivial. We get:

$$\frac{x-1}{5} = k, \quad \frac{y-4}{6} = k,$$

hence $6(x-1) - 5(y-4) = 0$, i.e.,

$$6x - 5y + 14 = 0, \tag{1}$$

which is the equation of a straight line, a consequence of the fact that both functions of k here are linear polynomials. More importantly, we note that (1) does not have any terms related to k.

Formally, we may write: $6x - 5y + 14 = 0$ is the locus of all points that move in space such that $x = 5k + 1$ and $y = 6k + 4$ for real values of k.

Example 2.

$$x = \tan\theta, \quad y = \cos\theta.$$

We recall the identity $\sec^2\theta - \tan^2\theta = 1$ and thus we get:

$$\frac{1}{y^2} - x^2 = 1.$$

This may be simplified further, but we have already got what we wanted: an equation in x and y, free of θ.

Also, in order to carry out the required parametrization, we require two key formulae, the proofs of which shall not be discussed here but which may be easily derived. For a projectile with velocity of projection v and angle of projection θ, the following hold (where g is the gravitational acceleration):

$$\text{Maximum height achieved by projectile} = \frac{v^2 \sin^2 \theta}{2g} \tag{2}$$

$$\text{Horizontal range of projectile} = \frac{v^2 \sin 2\theta}{g} \tag{3}$$

We are now ready to tackle the problem.

Solving the Problem

Consider the following projectile motion, where we have to trace the point MH (standing for 'Maximum Height') as it moves with a changing angle of projection.

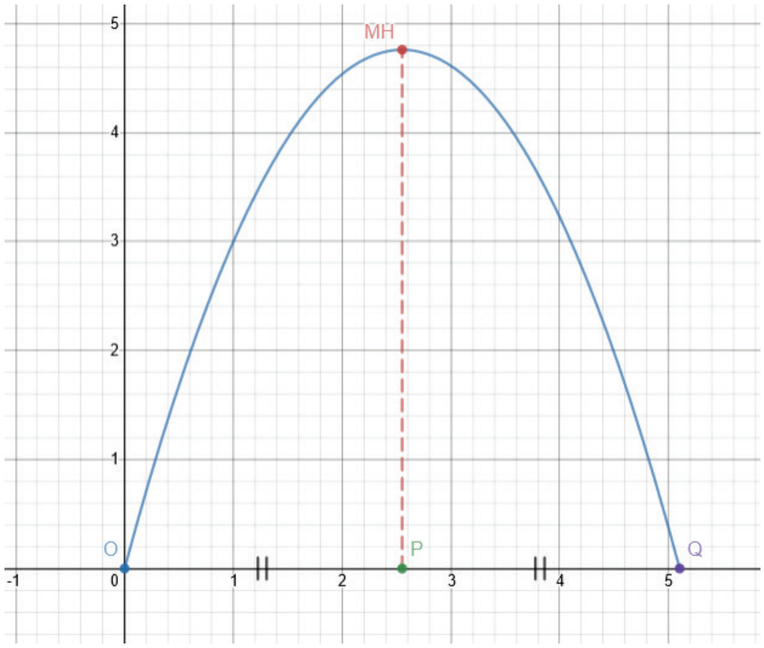

Figure 2. *MH* marks the Maximum Height

We have:

$$\text{Ordinate of } MH = \text{Maximum height achieved by projectile} = \frac{v^2 \sin^2 \theta}{2g}$$

$$\text{Abscissa of } MH = \frac{1}{2} \times \text{Horizontal range of projectile} = \frac{v^2 \sin 2\theta}{2g}.$$

Therefore, for a given v and θ,

$$B = (x, y) = \left(\frac{v^2 \sin 2\theta}{2g}, \frac{v^2 \sin^2 \theta}{2g} \right). \tag{4}$$

We now need to eliminate θ from the given parametrization:

$$x = \frac{v^2 \sin 2\theta}{2g}, \quad y = \frac{v^2 \sin^2 \theta}{2g}. \tag{5}$$

Notice that all we must do is find a relation between the trigonometric terms, i.e., the terms involving θ. Then, we can transpose and substitute the terms containing x and y as per our requirements. In this case, the terms are $\sin^2 \theta$ and $\sin 2\theta$. Now we have the relations

$$\begin{aligned} \sin 2\theta &= 2 \sin\theta \cos\theta, \quad \therefore \ \sin^2 2\theta = 4\sin^2\theta \cos^2\theta, \\ \therefore \ \sin^2 2\theta &= 4 \sin^2 \theta \left(1 - \sin^2 \theta\right). \end{aligned} \tag{6}$$

From (5),

$$\sin^2 2\theta = \left(\frac{2gx}{v^2}\right)^2, \quad \sin^2\theta = \frac{2gy}{v^2}. \tag{7}$$

Substituting, we get the following simplification. All cancellations are valid as all of the quantities, v, g, and θ, are non-zero.

$$\begin{aligned} &\sin^2 2\theta = 4\sin^2\theta\left(1 - \sin^2\theta\right) \\ \implies &\left(\frac{2gx}{v^2}\right)^2 = 4\left(\frac{2gy}{v^2}\right)\left(1 - \left(\frac{2gy}{v^2}\right)\right) \\ \implies &\frac{4g^2x^2}{v^4} = \left(\frac{8gy}{v^2}\right)\left(\frac{v^2 - 2gy}{v^2}\right) \\ \implies &gx^2 = (2y)(v^2 - 2gy) \\ \implies &gx^2 = 2yv^2 - 4gy^2. \end{aligned} \tag{8}$$

We can stop at (8) and say that this describes the path that the moving point traces. But we want to find out the shape, so we must manipulate it further.

Notice that as (8) is a second-degree equation in x and y, it represents either a conic section or a pair of straight lines. Let us thus continue:

$$\begin{aligned} &4y^2 - \frac{2yv^2}{g} + x^2 = 0 \\ \implies &y^2 - \frac{yv^2}{2g} + \frac{x^2}{4} = 0 \\ \implies &\frac{x^2}{4} + \left(y - \frac{v^2}{4g}\right)^2 = \left(\frac{v^2}{4g}\right)^2 \\ \implies &\frac{x^2}{4\left(v^2/4g\right)^2} + \frac{\left(y - v^2/4g\right)^2}{\left(v^2/4g\right)^2} = 1 \\ \implies &\frac{x^2}{\left(v^2/2g\right)^2} + \frac{\left(y - v^2/4g\right)^2}{\left(v^2/4g\right)^2} = 1. \end{aligned} \tag{9}$$

Et voilà, the equation of an ellipse! An ellipse, of all curves, has appeared unexpectedly.

The graph below clearly shows that the required figure is indeed an ellipse, because for all trajectories, MH_i for all $i = 1, 2, 3, 4, 5, 6$ always lies on the ellipse.

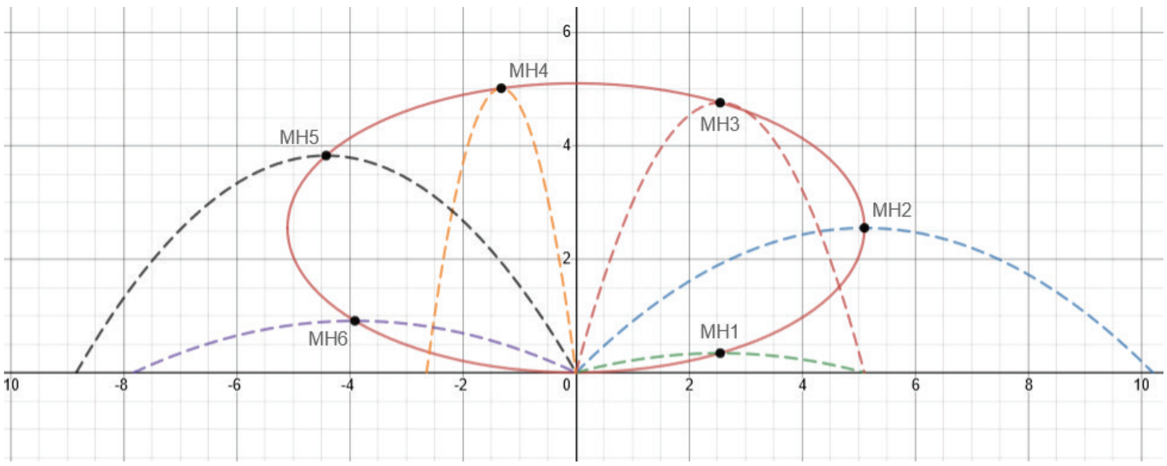

Figure 3. An ellipse!

The coordinates of the centre of the ellipse are

$$\left(0, \frac{v^2}{4g}\right),$$

and the lengths of the semi-major axis and the semi-minor axis are

$$\frac{v^2}{2g}, \quad \frac{v^2}{4g}.$$

Further questions to ponder

This problem isn't just an interesting exercise in itself; it is also a rich mine of further questions.

(1) What happens if instead of keeping v and g constant and varying θ, we keep g and θ constant and vary v? What is the figure so formed? What happens if we vary only the gravitational acceleration? (Although the last question quickly departs from reality, the problem is a thought experiment, and hence we need not be limited by reality!)

(2) Note that we have only considered the 2D-Cartesian plane here. What would happen, if the man was standing at the origin of a 3 dimensional Cartesian space and could throw the projectile in any vertical plane while keeping v and g constant? What 3D-figure would be formed? What would be its implicit equation in x, y and z?

References

1. Ben Orlin's highly successful and comedic blog online, https://mathwithbaddrawings.com/
2. The original question that this article addresses is found here: https://mathwithbaddrawings.com/2020/03/04/a-compendium-of-cool-internet-math-things/
3. The original tweet on twitter which asked this question: https://twitter.com/InertialObservr/status/1153356695474585600
4. Figure 1 and Figure 2: https://www.desmos.com/calculator/vjrvnab8md
5. Figure 3: https://www.desmos.com/calculator/4jwskh8onn The values of g, v and θ can be varied and its effects can be seen here.

SUKETU PATNI is a student in 10th grade in Delhi Public School, Navi Mumbai. He is passionate about math, especially Euclidean geometry, complex numbers and calculus. He has also written an article for the Indian Statistical Institute (ISI), about extending the ranges of circular functions to all real numbers. He may be contacted at suketublackpanther9@gmail.com.