Student Corner

# Counting some more Triangles

SUNDARRAMAN M

In the November 2016 issue of *At Right Angles*, I published an article, *Counting Triangles*, in which I discussed the following problem: *Count the number of triangles formed in a triangle if n segments are drawn from one vertex to its opposite side, and h segments are drawn from another vertex to its opposite side.*

This was a special case of a more general problem (stated below) which I left as an open question. Recently, I revisited this problem to see if I could show something interesting. In this article I provide a solution to the general problem and also show how the constraint can be done away with, i.e., show how to count the number of triangles even if there are concurrent segments inside the triangle.

The only prerequisite for reading this article is the *inclusion-exclusion principle* for three sets $A$, $B$, $C$:

$$|A \cup B \cup C| = |A|+|B|+|C|-|A \cap B|-|B \cap C| - |A \cap C|+|A \cap B \cap C|,$$

where $|X|$ denotes the cardinality of a set $X$.

The problem may be formally stated as follows:

**Problem.** Count the number of triangles formed when $n, h, k$ line segments are drawn from the vertices $A, B, C$ of a given triangle $ABC$ respectively to the opposite sides, assuming that no three of these $n+h+k$ line segments concur. (See Figure 1 for an example when $n = 1, h = 2, k = 3$.)

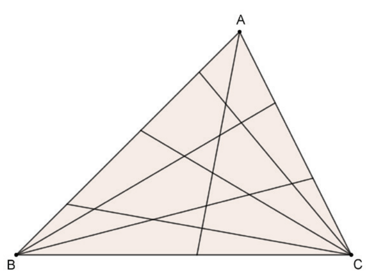


Figure 1.

*Keywords: inclusion-exclusion principle*

Now any triangle in Figure 1 must belong to one of the following two (disjoint) categories. 1) The triangle shares at least one vertex with $\triangle ABC$. 2) The triangle lies strictly in the interior of $\triangle ABC$.

Thus, the total number of triangles is equal to the number of triangles with at least one vertex in common with $\triangle ABC$, plus the number of triangles strictly in the interior of $\triangle ABC$.

It is relatively easy to count the number of triangles in the second category. For, each edge of such a triangle has to be one of the $n+k+h$ edges drawn from the vertices. Moreover, the triangle will have exactly one edge from among the $n$ edges drawn from $A$, exactly one edge from among the $h$ edges drawn from $B$ and exactly one edge from among the $k$ edges drawn from $C$. This is so because if there are two edges emanating from the same vertex, then that vertex will be a vertex of the triangle in question; and that is exactly what we wish to avoid. Thus the number of such triangles, if there are no concurrent edges, is $nkh$.

(Remark: In my earlier article, $h = 0$, which is why we didn't have this concept of internal triangles.)

So, the only thing left to calculate is the number of triangles with at least one vertex in common with $\triangle ABC$. This is where we use the inclusion-exclusion principle. Let the set of triangles containing vertex $A$ be denoted by $A$, the set of triangles containing vertex $B$ be denoted by $B$, and the set of triangles containing vertex $C$ be denoted by $C$. Then the number we need is $|A \cup B \cup C|$. By the inclusion-exclusion principle, we know

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|.$$

(Remark: Note that any triangle in this whole case is not dependent at all on the number of concurrent edges and so will remain the same in the slightly general case as well. We'll throw some light on this when we discuss the concurrent edges case.)

Consider first $|A|$. If you look at Figure 1, you will see that the 2 edges originating from $A$ will have to be among the $n+2$ edges originating from $A$. Once you have chosen the 2 edges, the third edge can be any edge not originating at $A$. This is because any such edge will intersect these 2 lines at unique points other than $A$ and thus will form a unique triangle. Thus, out of the $n+k+h+3$ edges in the figure, $k+h+1$ edges will each give a unique triangle (after removing the $n+2$ edges originating in $A$). Hence,

$$|A| = \binom{n+2}{2} \cdot (h+k+1).$$

The cardinalities $|B|$ and $|C|$ can be written using symmetry as follows:

$$|B| = \binom{h+2}{2} \cdot (n+k+1),$$

$$|C| = \binom{k+2}{2} \cdot (h+n+1).$$

Next, consider $|A \cap B|$. By definition, it is the number of triangles containing both $A$ and $B$ as vertices. This implies that edge $AB$ will be part of the triangle. If you look at Figure 1, you will notice that every triangle with $AB$ as an edge must have the remaining two edges originating from $A$ and $B$ respectively. Moreover, if you randomly select an edge originating at $A$ (other than $AB$) and an edge originating at $B$ (other than $AB$), you get a unique triangle. Therefore

$$|A \cap B| = (n+1) \cdot (h+1).$$

We have taken $n+1$ and $h+1$ and not $n+2$ and $h+2$ as we have excluded $AB$ from the selection. The other 2 cardinalities $|B\cap C|$ and $|C\cap A|$ can be written using symmetry as follows:

$$|B\cap C| = (k+1)\cdot(h+1),$$

$$|C\cap A| = (n+1)\cdot(k+1).$$

Finally, $|A\cap B\cap C| = 1$. (Why?)

Therefore the number of triangles in the figure is

$$\binom{n+2}{2}\cdot(h+k+1)+\binom{h+2}{2}\cdot(n+k+1)+\binom{k+2}{2}\cdot(h+n+1)$$
$$-(n+1)\cdot(h+1)-(k+1)\cdot(h+1)-(n+1)\cdot(k+1)+1+nkh.$$

So, this is a pretty clean formula. After simplification and some clever grouping, we get the following expression:

$$\frac{n^2(h+k+1)}{2}+\frac{h^2(n+k+1)}{2}+\frac{k^2(h+n+1)}{2}$$
$$+2(nh+hk+kn)+\frac{3(n+h+k)}{2}+1+nhk.$$

**Solution to a more generalized problem.** What happens when there are concurrent edges in the interior of the triangle? Note that it is not possible for more than three edges to concur at a point. Note also that each point of concurrence is like a triangle shrunk to a point, therefore every set of concurrent edges reduces the number of triangles by exactly 1.

From these observations, it follows that the number of triangles formed when $n, h, k$ line segments are drawn from the vertices $A, B, C$ respectively to the opposite sides and there are $t$ sets of concurrent segments, is equal to

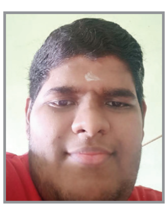
$$\frac{n^2(h+k+1)}{2}+\frac{h^2(n+k+1)}{2}+\frac{k^2(h+n+1)}{2}$$
$$+2(nh+hk+kn)+\frac{3(n+h+k)}{2}+1+nhk-t.$$

We close this article by asking the following question.

**Problem.** Count the number of triangles formed in a quadrilateral with $n_1, n_2, h_1, h_2, k_1, k_2, y_1, y_2$ edges from each of its vertices to points on the two non-adjacent sides respectively.

## References


1. Sundarraman M, "Counting triangles", *At Right Angles*, November 2016


**SUNDARRAMAN M** is a Mathematics and Computer Science graduate from Chennai Mathematical Institute. He is currently studying for his M Sc. His areas of interest include analytic number theory, computational number theory, statistics, and advanced algorithms. He may be contacted at sundarraman1607@gmail.com.