At Right Angles
A Resource for School Mathematics

NUMBER BASES

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NUMBER BASES

Students learn to count, read and write numbers in the decimal system for the first seven years of their schooling. They also learn how to read analogue clocks and use Roman numerals, and they may come across words such as feet, dozen and pound.

If they are to gain a good understanding and appreciation of the decimal base system, it is desirable that they are exposed to other number base systems like binary and hexadecimal, and do some operations in non-positional systems like the Roman system.

Through this exercise, students begin to understand the underlying structure of the positional system and its usage in other number bases. It helps them realise that the decimal base system is only one amongst other possible number systems. My experience of teaching number bases to students of class 6 has always been very rewarding, and I recommend it to all maths teachers of classes 6 and 7.

The Mohenjo-Daro culture of the Indus Valley was using a form of decimal numbering some 5,000 years ago as weights: 1/20, 1/10, 1/5, 1/2. The poet-mathematician Pingala (3rd/2nd century BCE) developed the binary number system for Sanskrit prosody, with a clear mapping to the base ten decimal system.

It is, of course, well known that the invention of zero as a number happened later in India. (The notion of zero did arise in some earlier cultures, but it served only as a placeholder, and it never entered into any arithmetical operations.)

The Roman system, which is not a positional system with its many symbols (I, V, X, L, C, D and M), is cumbersome and difficult to use as the numbers begin to grow in size. To write a million, one would have to use a thousand M's! Creating more and more symbols poses its own difficulties. Another major difficulty with the Roman system is the complexity of doing number operations. Try adding CLMDV to LDVXC, and you will see this for yourself!

Historically, vocations like carpentry and masonry needed facility with fractions. Wrenches are made in inches with half, quarter, eighth and sixteenths as their measures. Working with these measures and performing mathematical operations can be difficult, e.g., when fractions have to be multiplied. Yet, in many areas we continue to use fractions. Usage of fractions is evident in pizzas: a pizza is generally divided into 8 pieces. It would be difficult to divide a pizza into ten equal pieces!

In measurement of weight and volume, base 16 has often been used. An ounce equals 16 drams, a pound is 16 ounces, a cup equals 16 tablespoons, and a gallon is 16 cups.

Base 12 is familiar to us in the measurement of time, in cooking, etc. We often count items such as fruits and eggs in dozens. A dozen dozens is called a *gross*. And there are 12 months in a year. http://en.wikipedia.org/wiki/Duodecimal

In today's world, computers use the binary system as binary systems can be easily represented as on/off in electrical circuits. The input is either a zero or a one. This simplifies the information as there are only two states of representation. Computers also use groups of four digits, eight digits, and sixteen digits. Easy conversions between binary systems and octal or hexadecimal systems aid in the computations. However, we will not discuss this connection in this article.

While we have used varied systems through history, for everyday purposes we generally use the decimal system. One must see, however, that each system has its advantages and disadvantages. In some systems, the number representations may be very long (as there are too few symbols), while in some other systems, there may be too many symbols.

HOW DOES THE DECIMAL SYSTEM COMPARE WITH OTHER NUMBER BASED SYSTEMS?

Finger counting: The fact that human beings have ten fingers makes counting in tens easier. Counting in other bases using the fingers can prove to be difficult.

Length of representation: A binary number system leads to long representations of numbers. For instance, the base 10 number 365 is written in base two as 101101101. The same number in the hexadecimal system is written as 16D. These systems are explained further down in the article.

Number of symbols: If the number of symbols is large as in the case of the hexadecimal system (0 to 9, A, B, C, D, E, F), then one has to commit to memory several symbols and learn to handle many more operations. For example, we must know the values of C + F, D × E, and so on.

In daily life, one needs to choose a base for which representations are not too long, and at the same time the number of symbols is manageable. If we had to make a choice today, perhaps the decimal system will continue to be an obvious choice.

However, the binary, hexadecimal, and duodecimal systems have their rightful place in many areas.

In this article, we will explore the binary, hexadecimal and duodecimal (base twelve) systems.

Comment

I prefer to help students discover the underlying common structure by playing with and manipulating the numbers. As a second step I connect it with their understanding of writing numbers in exponential form and discovering the common structure. Others might prefer to explain the underlying common structure that holds for all number bases at the beginning and then develop the topic. I will leave this choice to the reader.

BASE 2: BINARY SYSTEM

The binary system has been used in different forms in the distant past, in ancient civilisations like Egypt, China and India. In the recent past, it was Leibniz and George Boole who studied these systems and worked on them.
https://en.wikipedia.org/wiki/Binary_number

The word 'Bi' means *two*. It is a system with two digits. Each digit is referred to as a bit (from binary and digit).

A single binary digit is called a 'bit'. 1001 is read as 'one, zero, zero, one'. It is four bits long.

Binary system or base two system implies the usage of two digits 1 and 0 in the framework of a positional system.

How do we represent different numbers in this system?

- Obviously, 1 (base ten) gets represented by 1, and a zero by 0.
- To represent 2 (base ten), a new place is needed; 2 is represented by 10.

The teacher can bring in the connection with the bundling idea that students have learnt in their early years. Ten units are bundled to make one ten which is represented in a new place as 10. Ten tens are bundled to make one hundred which is represented as 100.

Similarly, in the binary system, a set of two is bundled to make 10. Since 3 (base ten) is one more than 2 (base ten), 3 (base ten) is represented by 11.

4 (base ten) which is two sets of two is bundled to make a 100.

5 (base ten) is one more than 4 (base ten), hence is represented in base two by 101.

Decimal number	0	1	2	3	4	5	6	7	8	9	10	11	12
Binary number	0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100

The sequence would begin to look like this:

Students should be encouraged to build a table like this, in sequence, till 25 (base ten).

Also, they need to maintain the sequence; that is, after reaching (say) 1000, the whole process begins again, starting from the rightmost place: 1001, 1010, 1011, etc. Do the students see the similarity between this and the way decimal numbers increase: 101, 102, 103, 104…?

After writing numbers till 25 in binary form, the students should answer these questions:

At which points did we have to create a new place? What would be the next number that requires a new place? Check and see. It turns out to be 32.

Observe the pattern and determine what would be the next number after 32 that requires a new place. It is 64.

What can you say about these numbers?

2, 4, 8, 16, 32, 64, …

They are powers of 2. Each power of 2 becomes a transition point to a new place. Notice that in the decimal system too, each power of 10 becomes a transition point.

After 9, to write ten (10^1), we require the tens place (which is a bundle of 10 units). To write a hundred (10^2), we again require a new place (i.e. 10 bundles of ten units). Similarly, to write a thousand (10^3), we require a new place yet again (i.e. 10 bundles of hundred units).

Students can contrast the positional system of the Binary and Decimal numbers.

Students need to observe the similarities to understand the underlying structure of different based systems by studying the figure below. In a decimal system what would replace the numbers from 2^7 to 2^0?

What digits may appear instead of just 0 and 1?

Here is a decimal number: 59012.

Note: The word *coefficient* will need to be explained.

$59012 = 5 \times 10^4 + 9 \times 10^3 + 0 \times 10^2 + 1 \times 10^1 + 2 \times 10^0$

Look at the coefficients in this expansion. What are they? They are the digits 0, 1, 2, 5, 9. In a decimal number the coefficient can be any digit from 0 to 9.

Here is a binary number: 10110011.

$10110011 = 1 \times 2^7 + 0 \times 2^6 + 1 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$

Look at the coefficients in this expansion. What are they? They are the digits 0, 1. In a binary number the coefficients can be either 0 or 1.

What is common to the two expansions?

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	0	1	1	0	0	1	1
1×2^7		1×2^5	1×2^4			1×2^1	1×2^0
128		32	16			2	1

$$128 + 32 + 16 + 2 + 1 = 179$$

Build up a table of binary numbers. Shade all the 1's. Does it give rise to a nice pattern?

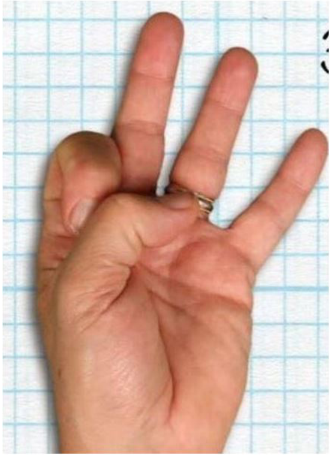
0	0	0	0
0	0	0	1
0	0	1	0
0	0	1	1
0	1	0	0
0	1	0	1
0	1	1	0
0	1	1	1
1	0	0	0
1	0	0	1
1	0	1	0
1	0	1	1
1	1	0	0
1	1	0	1
1	1	1	0
1	1	1	1

Fun activity

How many numbers can you count using your fingers? Most students will answer 'ten' as we have ten fingers.

Show the students how we can use our fingers to count till 31 using just one hand, and till 1023 using both hands!

Use the fingers as binary digits starting from the little finger. If the finger is down, it is a 0. If it is up, it is a 1. Each finger represents a power of 2 if it is up.



Thumb down, pointer finger down, middle up, ring finger up, little finger up. What number is that? 7.

Thumb down, pointer finger up, middle finger up, ring finger up, little finger down. What number is that? 14.

With ten fingers, it is possible to show all the numbers from 0 to 1023.

Can the students now convert a binary number to a decimal number? Their facility in doing this conversion will reveal their understanding of the structure. In the process of converting binary to decimal, they will use powers of 2 and expanded notation.

Would they be able to convert a decimal number to a binary number?

Figuring out the process of converting decimal to binary is an interesting investigation to try in the class. Will the students use the highest power of 2 less than the given number and work out the answer, step-by-step? Example: Take 1050. The power of 2 closest to and less than 1050 is 1024. When that is removed, the remainder is 1050 – 1024 = 26. The power of 2 closest to and less than 26 is 16. When that is removed, the remainder is 26 – 16 = 10. Then we have 8, leaving remainder 2. So the number 1050 in binary form is

$$1 \times 2^{10} + 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^1.$$

Therefore, 1050 (base ten) is 10000011010 in base two.

By writing a number as sum of powers of 2, students can study the polynomial structure of base two:

$$25 = 16 + 8 + 1 = 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0.$$

This can be connected to the division approach and the usage of the remainder at each stage.

$$\begin{aligned} 25 &= (2 \times 12) + 1 \\ &= 2 \times (2 \times 6) + 1 \\ &= 2 \times (2 \times (2 \times 3)) + 1 \\ &= 2^3 \times (2 + 1) + 1 \\ &= 2^4 + 2^3 + 2^0 \end{aligned}$$

Ex. To convert 25 (base ten) to binary, divide repeatedly by 2 and note the remainders.

Write the result bottom up. We get 11001:

2	25	
2	12	1
2	6	0
2	3	0
2	1	1
	0	1

OPERATIONS IN THE BINARY SYSTEM

Students can explore addition and subtraction in the binary system to understand the usage of carry over, exchange with fours, twos, etc.

What is of interest is to see the same mechanism at work as in the familiar decimal system.

This can be demonstrated through a few problems requiring addition and subtraction operations.

$$
\begin{array}{r}
\color{red}{1\,1} \\
0\,1\,0\,1\,0 \\
+\ \ 1\,1\,1\,0\,1 \\
\hline
1\,0\,0\,1\,1\,1
\end{array}
$$

What is 0 + 1? What does 1 + 1 become? What digit goes as a carry over?

$$
\begin{array}{r}
1\,\cancel{1}^{\color{red}{0}}\,\cancel{0}^{\color{red}{10}} \\
-\ \ 1\,0\,1 \\
\hline
1
\end{array}
$$

In the subtraction problem, why has a 10 replaced the zero in the right most place?

Are these addition and subtraction methods similar to the methods of the decimal system?

Do commutativity, associativity, transitivity and distributivity laws hold for binary numbers?

MULTIPLICATION

How does multiplication work out in the binary system? Is there an identity element in binary numbers?

Here is a sample multiplication problem in the binary system.

$$
\begin{array}{rrrrr}
 & & 1 & 1 & 0 \\
\times & & & 1 & 1 \\
\hline
 & 1 & 1 & 1 & 0 \\
 & 1 & 1 & 0 & \times \\
\hline
1 & 0 & 0 & 1 & 0 \\
\hline
\end{array}
$$

What are the possible multiplications that can arise in a binary system?

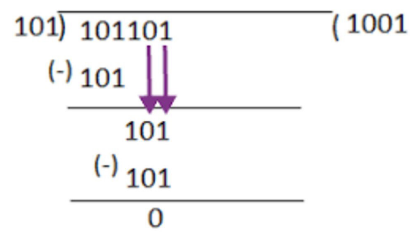
$$0 \times 0 = 0$$
$$0 \times 1 = 0$$
$$1 \times 1 = 1$$

So much simpler to memorise these tables!

Here is a sample binary division problem. In decimal terms, it would be the same as 45 divided by 5 which equals 9.

Does the usual division method work in the binary system? Explore and see.

Are there fractional numbers in the binary system? What would .1 mean in the binary system?



Just as the numbers to the left of a decimal point in the decimal system are whole numbers, the numbers to the left of a decimal point in the binary system too represent whole numbers.

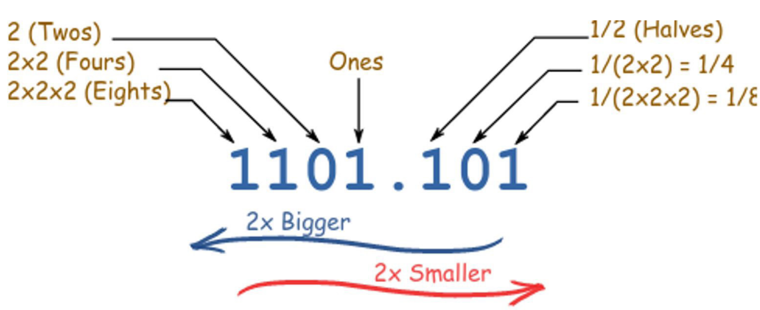
As we move to the left, with each step, place values get doubled (i.e., they are twice as big). As we move to the right, with each step, place values get halved (i.e., they are half as big). Hence the first digit on the right (.1) means 'half', and .01 means 'quarter'.

How does one identify a number as binary? In order to indicate the base, the practice followed is to show the base as a subscript. For example, 1011_2

Investigation

Do the binary representations of all rational numbers terminate? Or do some recur?

Binary positional system



BASE 16: HEXADECIMAL NUMBERS

The word 'Hexadecimal' means 'based on 16.' The value of the base in the hexadecimal system is 16.

What are the basic digits in the hexadecimal system?

Apart from 0 to 9, 10 is represented by A, 11 by B, 12 by C, 13 by D, 14 by E, and 15 by F.

Decimal number	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Hexadecimal number	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10
Decimal number	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
Hexadecimal number	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	1F	20	21

Let students experiment with hexadecimal numbers by converting hexadecimal numbers to decimal numbers, and decimal numbers to hexadecimal numbers.

Let them build the positional table for a hexadecimal system and use it for conversions.

16^3	16^2	16^1	16^0
1	*B*	7	*E*
1×16^3	11×16^2	7×16^1	14×16^0
4096	2816	112	14

$$4096 + 2816 + 112 + 14 = 7038$$

What would the value of .1 be in the hexadecimal system?

In the hexadecimal system, as we move left, each place is 16 times bigger. The first number to the right of the point is one-sixteenth. As we move right, each place is 16 times smaller.

How would fractional numbers like 1/2 or 1/4 get represented in this system?

How would negative numbers get represented?

Students can check whether the procedures used for the binary system conversions work in this setting. How will they adapt the procedures to suit this system?

Here is a sample division procedure to generate the hexadecimal number for 447. The result is read bottom up as 1BF. (Remember that the symbols in this system are the digits 0 to 9 and A, B, C, D, E, F.)

Division	Quotient	Remainder
$447 \div 16$	27	$15 = F$
$27 \div 16$	1	$11 = B$
$1 \div 16$	0	$1 = 1$

Here are sample addition and subtraction problems for hexadecimal numbers. (The digits shown in red are the 'carry-overs'.)

Sample addition problem	Sample subtraction problem
$\begin{array}{cccccc} & 1 & 1 & & & \\ & & 7 & F & B & 3 \\ + & 1 & B & 6 & 2 & 1 \\ \hline & 2 & 3 & 5 & D & 4 \end{array}$	$\begin{array}{cccccc} & & E & 21 & 6 & 26 \\ & 3 & F & 5 & 7 & A \\ - & & C & 8 & 5 & E \\ \hline & 3 & 2 & D & 1 & C \end{array}$

What does a carry-over of 1 represent in this system?

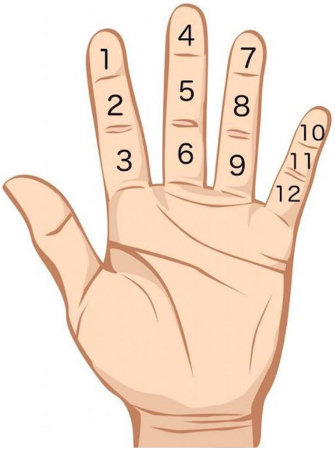
In the subtraction problem, why is there a 26 over A?

Are these addition and subtraction methods similar to the methods of the decimal system? In what way are they similar? In what way are they different?

BASE 12: DUODECIMAL SYSTEM

Humankind has had a strong connection since ancient times with base 12, and it would be a pity if students do not get to study it.

The analogue clock has twelve hours displayed on its face. The number of hours in a day (24) is a multiple of 12. An hour has sixty minutes which is a multiple of 12. The number of degrees in a circle is 360 which again is a multiple of 12.



The segments on the four fingers are 12 in number, and can be used to count in base 12. It is said that the Babylonians used to count the three segments of their four fingers to get 12. They marked that 12 by raising a finger on the other hand. This way they could count up to 60 (twelve times five fingers being 60). Now 60 has excellent properties, being divisible by 2, 3, 4, 5, 6, 10, 12, 15, 20, 30 and 60 itself. This means that fractions will not pose too many difficulties!

https://www.earthdate.org/how-10-fingers-became-12-hours

Decimal number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Duodecimal number	1	2	3	4	5	6	7	8	9	A	B	10	11	12	13	14	15	16	17	18

Students can create a table of addition facts for base 12 as an extension in their study of operations in base 12.

	0	1	2	3	4	5	6	7	8	9	A	B
0	0	1	2	3	4	5	6	7	8	9	A	B
1	1	2	3	4	5	6	7	8	9	A	B	10
2	2	3	4	5	6	7	8	9	A	B	10	11
3	3	4	5	6	7	8	9	A	B	10	11	12
4	4	5	6	7	8	9	A	B	10	11	12	13
5	5	6	7	8	9	A	B	10	11	12	13	14
6	6	7	8	9	A	B	10	11	12	13	14	15
7	7	8	9	A	B	10	11	12	13	14	15	16
8	8	9	A	B	10	11	12	13	14	15	16	17
9	9	A	B	10	11	12	13	14	15	16	17	18
A	A	B	10	11	12	13	14	15	16	17	18	19
B	B	10	11	12	13	14	15	16	17	18	19	1A

Multiplication in base 12 produces nice patterns and makes for an interesting project.

1	2	3	4	5	6	7	8	9	A	B	10
2	4	6	8	A	10	12	14	16	18	1A	20
3	6	9	10	13	16	19	20	23	26	29	30
4	8	10	14	18	20	24	28	30	34	38	40
5	A	13	18	21	26	2E	34	39	42	47	50
6	10	16	20	26	30	B	36	40	46	50	60
7	12	19	24	2E	B	41	48	53	5A	65	70
8	14	20	28	34	40	48	54	60	68	74	80
9	16	23	30	39	46	53	60	69	76	83	90
A	18	26	34	42	46	5A	68	76	84	92	A0
B	1A	29	38	47	50	65	74	83	92	A1	B0
10	20	30	40	50	60	70	80	90	A0	B0	100

To study and contrast different number systems is enjoyable and stimulating for the students of an upper primary school.

CLOSING REMARK

In comparing base ten with other bases, there are two other factors which we did not consider earlier.

- **Tests of divisibility:** In base ten, divisions by 2 and 5 are very easy to carry out. This is because both 2 and 5 are divisors of 10. For the same reason, the tests for divisibility by 2 and by 5 are easy to understand and carry out in base ten. In general, for a given base *b*, the tests of divisibility by different numbers are easy to carry out when the numbers are divisors of *b* (or divisors of powers of *b*). Base 12 (duodecimal system) scores well in this sense, as 12 has many divisors (proper divisors: 2, 3, 4 and 6).
- **Terminating decimals:** In base ten, the decimal form of a fraction terminates when the denominator of the fraction is a product of powers of 2 and 5. If not, the decimal form is non-terminating (it recurs). One disadvantage of the binary system is that comparatively few fractions (those whose denominators are powers of 2) terminate. E.g., 1/10 does not have a terminating binary representation.

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