In the TearOut for the November 2018 issue (page 43 Fun with Dot Sheets) Swati Sircar had ended with a tantalizing observation and the promise of a proof. The observation was: Note that one type of triangle (with all three vertices on lattice points) is not possible. This is the equilateral triangle - An equilateral triangle can't be drawn with lattice points as vertices. Elsewhere in this issue, Shailesh Shirali has provided four proofs of the same result. Here is one more from Swati!

We will do this in two parts:

1. Triangle with one side horizontal or vertical
2. Triangle with no side horizontal or vertical

   a. This is quite easy using Pythagoras and the fact that for an equilateral triangle with base b and height h, $h = \left(\frac{\sqrt{3}}{2}\right)$ b. Let's assume without loss of generality that one side is horizontal. Again without loss of generality we can take the vertices on this side to be $(-n, 0)$ and $(n, 0)$ where n is a natural number. So the 3rd vertex will be of the form $(0, m)$ where m is a natural number and is the height of the triangle. So $m = \frac{\sqrt{3}}{2} \times 2n = \sqrt{3}n$ which is irrational. Similarly, if one side is vertical we can take the vertices as $(0, n)$ and $(0, -n)$ and $(m, 0)$ and again get $m = \sqrt{3}n$.

   So it is not possible to locate the third vertex on a lattice point if one side is either horizontal or vertical.

   b. If an equilateral triangle does not have any side horizontal or vertical, then each side is slant i.e. each has a non-zero slope. So the triangle can be translated along the axes if needed to get two of its vertices lie of the two axes. We can take these two vertices to be M = $(m, 0)$, N = $(0, n)$ and let the 3rd vertex be L (x, y). Since we are talking about lattice points, m, n, x and y are non-zero integers. Let P be the midpoint of MN i.e. P = $\left(\frac{m}{2}, \frac{n}{2}\right)$. Now, LP ⊥ MN and LP = $\frac{\sqrt{3}}{2}$ × MN. This gives us two equations:

   $$\frac{y-\frac{n}{2}}{x-\frac{m}{2}} = \frac{m}{n} \Rightarrow y - \frac{n}{2} = \frac{m}{n}\left(x - \frac{m}{2}\right) \text{ …. (1) and}$$

   $$\left(x-\frac{m}{2}\right)^2 + \left(y-\frac{n}{2}\right)^2 = \frac{3}{4}(m^2+n^2) \text{ …. (2)}$$

   Using (1) and substituting for $y - \frac{n}{2}$ in (2) and simplifying, we get $\left(x - \frac{m}{2}\right)^2 = \frac{3}{4}(n^2)$

   $\Rightarrow x = \frac{m}{2} \pm \left(\frac{\sqrt{3}}{2}\right)n$ and therefore $y = \frac{n}{2} \pm \left(\frac{\sqrt{3}}{2}\right)m$

   Since m and n are non-zero, x and y are irrational. Therefore L can't be a lattice point regardless of our choice of m and n.

   It is interesting to note the symmetry in the coordinates of an equilateral triangle of this kind $(m, 0)$, $(n, 0)$ and $\left(\frac{m}{2} \pm \left(\frac{\sqrt{3}}{2}\right) n, \frac{n}{2} \pm \left(\frac{\sqrt{3}}{2}\right) m\right)$.