# Searching for Pythagorean Quadruples

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A *Pythagorean quadruple* is a 4-tuple of positive integers $a, b, c, d$ such that

$$a^2 + b^2 + c^2 = d^2. \tag{1}$$

In this article we will try to find a way to generate Pythagorean quadruples.

Let $d = a + m$. Since $(a + m)^2 = a^2 + 2am + m^2$, if we can find integers such that

$$b^2 + c^2 = 2am + m^2, \tag{2}$$

then the relation $a^2 + b^2 + c^2 = d^2$ will be satisfied.

We consider separately the cases when $b^2 + c^2$ is odd and when it is even.

**Case 1: $\mathbf{b^2 + c^2}$ is odd.** Then one of $b^2$ and $c^2$ is odd and other is even.

From (2) we get $2am + m^2 = b^2 + c^2$, hence

$$am = \frac{b^2 + c^2 - m^2}{2}. \tag{3}$$

Since $b^2 + c^2$ is odd, by assumption, $m^2$ and therefore $m$ is odd.

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Next we have $d = a + m$, so $dm = am + m^2$, and so

$$dm = \frac{b^2 + c^2 - m^2}{2} + m^2 = \frac{b^2 + c^2 + m^2}{2}. \tag{4}$$

The quantity on the right side is not necessarily a multiple of $m$, so we scale up the 4-tuple $(a, b, c, d)$ by a factor of $m$. That is, we consider instead the 4-tuple $(am, bm, cm, dm)$. We may now generate infinitely many such 4-tuples $(am, bm, cm, dm)$ using the identity

$$\left(\frac{b^2 + c^2 - m^2}{2}\right)^2 + (mb)^2 + (mc)^2 = \left(\frac{b^2 + c^2 + m^2}{2}\right)^2. \tag{5}$$

**Example 1.** Take $b = 2$, $c = 5$, $m = 3$. Then (5) yields:

$$10^2 + 6^2 + 15^2 = 19^2.$$

**Example 2.** Take $b = 3$, $c = 8$, $m = 5$. Then (5) yields:

$$24^2 + 15^2 + 40^2 = 49^2.$$

**Case 2: $\mathbf{b}^2 + \mathbf{c}^2$ is even.** Essentially the same analysis works here too; we use identity (5) again. The only difference is that $m$ must now be an even integer. If both $b$ and $c$ are even, then there will be a common factor which can be divided out from the 4-tuple.

**Example 3.** Take $b = 3$, $c = 7$, $m = 4$. Then (5) yields:

$$21^2 + 12^2 + 28^2 = 37^2.$$

**Example 4.** Take $b = 3$, $c = 11$, $m = 8$. Then (5) yields:

$$33^2 + 24^2 + 88^2 = 97^2.$$

**Example 5.** Take $b = 4$, $c = 14$, $m = 10$. Then (5) yields:

$$56^2 + 40^2 + 140^2 = 156^2.$$

We see that all the terms have a common factor of 4 which we may divide out. This yields:

$$14^2 + 10^2 + 35^2 = 39^2.$$

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