6 Number: The Role of Pattern and Play in its Teaching

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Pre-history

The concept of number is crucial to Mathematics, yet its origin may forever be hidden from us, for it goes far back in time. Human beings must have started long back to use the tally system for keeping records – livestock, trade, etc – but we may never know just when. A remarkable discovery made in 1960 in Belgian Congo of the Ishango bone, dated to 20,000 years BP, suggests that the seeds of Mathematical thinking may go still further back than thought; for, carved on the bone are tally marks grouped in a deliberate manner, seemingly indicative of a Mathematical pattern (there is even a hint of a doubling sequence: 2, 4, 8). However, until further evidence is uncovered, the matter must remain as speculation. See the Wikipedia reference for more information.

Tally counting as a practice may well be as many as 50,000 years old; even today we use it to count, in various contexts, e.g. in a class election.

Base ten number system

The notation we use today – the base ten or decimal system – has its origins in ancient practices. Long back the Babylonians used a system based on powers of 60, and traces of that practice remain to this day – we still have 60 seconds in a minute, 60 minutes in an hour, 60 minutes in a degree (for angular measure). Later the Egyptians developed a system based on powers of 10, in which each power of ten from ten till a million was represented by its own symbol. But this system differs from ours in a crucial way – it lacks a symbol for zero.

A system of arithmetic without a symbol for zero suffers from two difficulties. The first is that there is confusion between numbers like 23, which represents 2 tens and 3 units, and 203, which represents 2 hundreds and 3 units. Without the zero symbol some way has to be found to indicate that the 2 means “2 hundreds” and not “2 tens”. This can be done, but it is quite cumbersome. But a greater difficulty is that computations become significantly harder, and it becomes that much more difficult to progress in arithmetic.

The Greeks did not have a symbol for zero, and it is not surprising that they did not develop arithmetic and algebra the way they developed geometry, which they took to great heights. It was in India that the symbol for zero came into being (probably as early as the 5th century), along with the rules for working with it. Not coincidentally, arithmetic and algebra grew in a very impressive manner in India, in the hands of Aryabhata, Brahmagupta, Mahavira, Bhaskaracharya II, and many others.

On the other hand the ancient Indians did not progress anywhere as far in their study of geometry. But it is striking that one area where the methods of algebra and analysis enter into geometry in a natural way, namely trigonometry, did originate in India (in the work of Aryabhata, 5th century AD).

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Abstraction and the number concept

Embedded in our brains is an extraordinary ability: the ability to form concepts; the ability to abstract common features and shared qualities from collections of objects or phenomena. It is this ability that lies behind the creation of language, and it is this that enables us to “invent” numbers. To understand what this means, think of a number, say 3. Is 3 a thing? Can it be located somewhere? No, it cannot; but our brains have the ability to see the quality of “threeness” in collections of objects: three fingers, three birds, three kittens, three puppies, three people – the feature they share is the quality of threeness. This ability is intrinsic to the very structure of our brains. Were it not there, we would never be able to learn the concept of number (or any other such concept, because any concept is essentially an abstraction).

Even in something as simple as tally counting – creating a 1-1 correspondence between a set of objects and a set of tally marks – our brains show an innate ability for abstraction: by willfully disregarding the particularities of the various objects and instead considering them as faceless entities.

Realization of this insight has pedagogical consequence; for, as has been wisely said, "Concepts are caught, not taught". It is only by actual contact with collections of objects that concepts form in one's brain. How exactly this happens is still not well understood, but I recall a comment which goes back to Socrates *(the teacher's role is akin to that of a midwife who assists in delivery).*

The invention of algebra represents one more step up the ladder of abstraction. To illustrate what this means, let us examine these number facts: 1+3 = 4, 3+5 = 8, 5+7 = 12, 7+9 = 16, 9+11 = 20. We see a clear pattern: the sum of two consecutive odd numbers is always a multiple of 4. This statement cannot be verified by listing all the possibilities, for there are too many of them – indeed, infinitely many. But we can use algebraic methods! We only have to translate the observation into the algebraic statement (2n-1) + (2n+1) = 4n; this instantly proves the statement. Such is the power of algebra and also the power of abstraction – and this ability too is intrinsic to our brains.

Number patterns

Another feature intrinsic to the brain is the desire and capacity for play. Most mammals seem to have it, as we see in the play patterns of their young ones – and what a pleasing sight it can be, to watch kittens or puppies or baby monkeys at play! But human beings have a further ability: that of bringing patterns into their play. When our love of play combines with the number concept and with our love of patterns, Mathematics is born. For Mathematics is essentially the science of pattern.

It is crucial to understand the element of play in Mathematics; for one is told, repeatedly, of the utility ofMathematics, how it plays a central role in so many areas of life, and how it is so important to one's career. But the element of play gets passed over in this viewpoint; the subject becomes something one must know, compulsorily, and the stage is set for a long term fearful relationship with the subject.

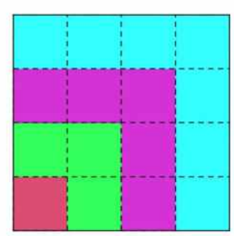
From the earliest times – in Babylon, Greece, China, India – there has been a playful fascination with number patterns and geometrical shapes one can associate with numbers. From this are born number families – prime numbers, triangular numbers, square numbers, and so on.

Let us illustrate what the term "pattern" means in this context. We subdivide the counting numbers 1, 2, 3, 4, 5, 6, 7, 8,... into two families, the odd numbers (1, 3, 5, 7, 9, 11, ...), and the even numbers (2, 4, 6, 8, 10, 12, ...). If we keep a running total of the odd numbers here is what we get: 1, 1+3 = 4, 1+3+5 = 9, 1+3+5+7 = 16, 1+3+5+7+9 = 25. Well! We have obtained the list of perfect squares!

There is a wonderful way we can show the connection between sums of consecutive odd numbers and the square numbers; it is pleasing to behold and incisive in its power at the same time. All we have to do is to examine the picture below: this property is closely related to one about the triangular numbers: the sequence 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, ... formed by making a running total of the counting numbers: 1, 1+2 = 3, 1+2+3 = 6, 1+2+3+4 = 10, etc. They are so called because we can associate triangular shapes with these numbers.

There is just one red square; when we put in three green squares around it, they together make a 2 by 2 square; hence we have 1 + 3 = 2 times 2.

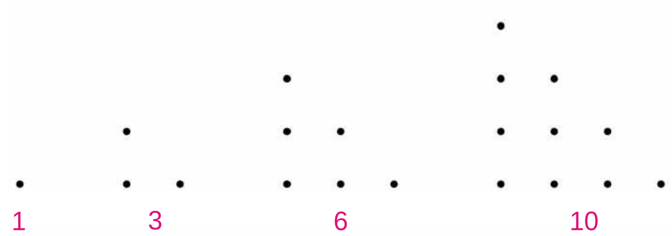
Put in the five purple squares and now you have a 3 by 3 square; hence 1 + 3 + 5 = 3 times 3.



Put in the seven blue squares, and now you have a 4 by 4 square; hence 1 + 3 + 5 + 7= 4 times 4. And so on!

There are two striking properties that connect the triangular numbers with the square numbers (1, 4, 9, 16, etc), and they can easily be found by children: (I) the sum of two consecutive triangular numbers is a square number; e.g., 1+3 = 4, 3+6 = 9, 6+10 = 16, ...; (II) if 1 is added to 8 times a triangular number we get a square e.g., (8 × 3) + 1 = 25, (8 × 6) + 1 = 49, (8 × 10) + 1 = 81.

Why is there such a nice connection? A lovely question to ponder over, isn't it?



Here is another pattern. Take any triple of consecutive numbers; say 3, 4, 5. Square the middle number; we get 16. Multiply the outer two numbers with each other; we get 3 times 5 which is 15. Observe that 16 – 15 = 1; the two numbers obtained differ by 1. Try it with some other triple, say 7, 8, 9: 8 squared is 64, 7 times 9 is 63, and 64 – 63 = 1; once again we get a difference of 1. Will this pattern continue? Yes, and it is easy to show it using algebra; but think of what pleasure discovering this can give a young child playing with numbers!

We find a similar but more complex pattern with the famous Fibonacci sequence, which goes 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...; here, each number after the first two is the sum of the preceding two numbers (e.g., 8 = 5 + 3). Repeat the computation with this sequence. With the triple 2, 3, 5 we get: 3 squared is 9, and 2 times 5 is 10; the squared number is 1 less than the product of the other two. With the triple 3, 5, 8 we get: 5 squared is 25, and 3 times 8 is 24; now the squared number is larger by 1. With 5, 8, 13 we get: 8 times 8 is 64, and 5 times 13 is 65; once again the squared number is smaller by 1. And so it goes – a curious alternating pattern.

We see the same thing if we study collections of four consecutive Fibonacci numbers; for example, 1, 2, 3, 5. The product of the outer two numbers is 5, and the product of the inner two is 6; they differ by 1. Take another such collection: 3, 5, 8, 13. The product of the outer two is 39, that of the inner two is 40; once again, a difference of 1. And again the alternating pattern continues. Astonishingly, even nature sees fit to use the Fibonacci numbers. If we keep records of the numbers of petals in various flowers, we find that the number is generally a Fibonacci number. Study the spirals in which pollen grains are arranged in the center of a sunflower; there are spirals running in clockwise and anticlockwise directions; you will find that the number of spirals of each kind is a Fibonacci number. Nature is just as fond of patterns as we are!

Many years back I used a textbook called “Pattern and Power of Mathematics”. It is a nice title for a textbook, for patterns are what the subject is all about, and it is this that gives it its astonishing power. But – more important – it is this feature that makes us study the subject in the first place.

Large numbers, small numbers

There are numbers, and then there are large numbers. Children naturally like large numbers, and many of them discover on their own that there is no last number: however large a number one may quote, one only needs to add 1 to it to get a larger number. So the number world has no boundary! There are some who make a similar discovery at the other end – with small numbers; I recall a student telling me, many years back, how she could make an unending sequence of tinier and tinier fractions, simply by halving repeatedly; she could not believe that such tiny numbers could exist! She had made this wonderful discovery herself, and was very excited by it.

The ancient Indians loved large numbers, and here's a problem that shows this love. If I ask you to find a squared number that is twice another squared number, you would never succeed, because there aren't any such pairs of numbers. (Why? – there's a nice story behind that, but we cannot go into that now.) So we change the problem a little bit: I ask for a squared number which exceeds twice another squared number by 1. Now we find many solutions; e.g., 9 and 4 are squared numbers, and 9 – (2 × 4) = 1. Here are some more solutions:

289 – (2 × 144) = 1,
9801 – (2 × 4900) = 1.

If we replace the word “twice” by “5 times” we find solutions to this too:

81 – (5 × 16) = 1,
(161 × 161) – (5 × 72 × 72) = 1,
and so on.

In the 7^{th} century, Brahmagupta asked if we could find solutions with “5 times” replaced by “61 times”. The smallest solution in this case is very large indeed – yet Brahmagupta found it:

(1766319049 × 1766319049) - (61 × 226153980 × 226153980) = 1.

Feel free to verify the relation.

I think the date is significant: the Indians were asking such questions thirteen centuries back! The love of play has been there in all human cultures, for a long time. There's no holding it back.

But now a strange thing happens. What began as play takes wing, and flies away a mature subject, with an inner cohesiveness and structure that is strong enough to find application in the world of materials, living bodies, and finance – the “real world”. Such flights have happened two dozen times or more in history, and no one really knows how and why they happen; but they do. Maybe it is God's gift to us. (But we do not always use it as intended; the power of Mathematical methods also finds application in the design of bombs and nuclear submarines and other instruments of killing.)

Closing note

There are so many topics in which we can bring out the theme of pattern and play in Mathematics:

- Magic squares (arranging a given set of 9 numbers in a 3 by 3 array, or 16 numbers in a 4 by 4 array, so that the row sums, column sums, diagonal sums are all the same); not only do these bring out nice number relationships, but in the course of the study one learns about symmetry.

- Cryptarithms (solving arithmetic problems in which digits have been substituted by letters; for example, ON + ON + ON + ON = GO; many simple but pleasing arithmetical insights emerge from the study of such problems);

- Digital patterns in the powers of 2 (list the units digits of the successive powers of 2; what do you notice? Now do the same with the powers of 3; what do you notice?)

These examples are woven around the theme of number, but the principle extends to geometry in an obvious way. Here we study topics like rangoli and kolam; paper folding; designs made with circles; and so on.

Alongside such activities, teachers could also raise questions relating to the role of Mathematics in society, for discussion with students and fellow teachers; e.g., questions relating to the use of Mathematics for destructive purposes, or more generally, "When is it appropriate to use Mathematics?"; or the question of why society would want to support mathematical activity. After all, most artists find patrons or buyers for their art work, but mathematicians do not sell theorems for a living! Is it that policy makers see Mathematics as a useful tool, and thus enable people in this field to sustain themselves, by teaching or doing useful Mathematics? The notion of usefulness takes us back to the question of appropriateness of usage. Such questions are not generally seen as fitting into a Mathematics class, but there is clearly a place for them in promoting a culture of discussion and inquiry.

We need not try to make a complete listing here – it is not possible, because it is too large a list, and ever on the increase. Instead, we wish only to emphasize here that pattern and play are crucial to the teaching of Mathematics, for pedagogic as well as psychological reasons.

A great opportunity is lost when we make Mathematics into a heavy and serious subject reserved for the highly talented, and done under an atmosphere of heavy competition. It denies the experience of Mathematics to so many.

Suggested Reading

1. http://en.wikipedia.org/wiki/Ishango_bone
2. “Number, The Language Of Science” Tobias Dantzig
3. “The Number Sense: How the Mind Creates Mathematics” Stanislas Dehaene
4. http://en.wikipedia.org/wiki/History_of_Mathematics

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